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**Connecting the Provable with the Unprovable:  
Phase Transitions in Unprovability**

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*For my grandmothers.*



*I do not understand unprovability.*

Andrey Bovykin

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## Preface

Why are some theorems not provable in certain theories of mathematics? Why are most theorems from existing mathematics provable in very weak systems? Unprovability theory seeks answers for those questions. Logicians have obtained unprovable statements which resemble provable statements. These statements often contain some condition which seems to cause unprovability, as this condition can be modified, using a function parameter, in such a manner as to make the theorem provable. It turns out that in many cases there is a *phase transition*: By modifying the parameter slightly one changes the theorem from provable to unprovable.

We study these transitions with the goal of gaining more insights into unprovability.



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## Chapter 1

# Introduction

### 1.1 Unprovability

Unprovability theory, incompleteness phenomena, independence results, are three names by which this field is known. It is the research of theorems which are not provable in theories of mathematics. This phenomenon has been known since Gödel's incompleteness theorems, *as soon as a consistent system of axioms is sufficiently complicated, though still computable, there will be theorems which are not provable within this system.*

Gödel's results answer a part of problem number 2 from Hilbert's famous speech on Mathematical Problems also known as Hilbert's programme: To set up an axiom system of mathematics and, most importantly:

To prove that they are not contradictory, that is, that a finite number of logical steps based upon them can never lead to contradictory results.<sup>1</sup>

Whilst interesting in itself this has little implications in areas of mathematics outside of logic. The question was whether, given Peano Arithmetic, which was intended as an axiomatisation capturing all truths concerning arithmetic, there exists a *natural* theorem of arithmetic which is not provable in it. The first such examples arrived in 1977. Paris used the notion of Ramsey densities to obtain an independent statement which Harrington modified into the more elegant Paris–Harrington theorem [35], which has a strong flavour of Ramsey theory:

**1 Theorem** (Paris–Harrington, PH). *For every  $d, c, m$  there exists an  $R$  such that for every colouring  $C: [m, R]^d \rightarrow c$  there exists an  $H \subseteq [m, R]$  of size  $\min H$  for which  $C$  limited to  $[H]^d$  is constant.*

This theorem is proved, as is often done for the ordinary Ramsey theorem, using a compactness argument on the infinite version of Ramsey's theorem (See, for example, [20] or [33]). It is shown not to be provable in PA by giving a counterexample: A model of PA in which this theorem is not true. The Paris–Harrington theorem is regarded as natural in the sense that it closely resembles theorems found in a mathematics textbook (replace  $\min H$  by  $m$  to obtain the classical Ramsey theorem, which is provable even in weak fragments of PA) and requires no special

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<sup>1</sup>English translation by M. W. Newson in [21], page 414

definitions to state concisely. The Paris–Harrington theorem is also finite: It is arithmetical with very few unbounded quantifiers, so its strength does not derive from the use of infinite objects or ones which cannot be stated in the language of PA. More formally we are interested in natural theorems which are  $\Pi_2$ . In general natural finite theorems from the mathematics literature are provable in very weak axiom systems, for example exponential function arithmetic, which axiomatises addition, multiplication and exponentiation and has a very limited form of induction. Shortly after the discovery of PH more such natural unprovable statements emerged. We state some of the more remarkable ones. The second example related to Ramsey theory is [23]:

**2 Theorem** (Kanamori–McAloon, KM). *For every  $d, m$  there exists  $R$  such that for every colouring  $C: [R]^d \rightarrow \mathbb{N}$  with  $C(x) \leq \min x$  there exists  $H \subseteq R$  of size  $m$  for which for all  $x, y \in [H]^d$  with  $\min x = \min y$  we have  $C(x) = C(y)$ .*

In 1944 Goodstein [19] introduced an infinite proposition about the natural numbers which is equivalent to the consistency of Peano Arithmetic. Kirby and Paris [25] showed that a finite version is not provable in PA:

**3 Theorem** (Goodstein sequences). *Examine the following process: Starting with  $n = n_0$ , in step  $i$  replace in the representation of  $n_i$ , written in complete base  $i + 2$  representation, every occurrence of  $i + 2$  with  $i + 3$  and subtract 1 from the resulting number. This process reaches 0 for every  $n$ .*

In the same article Kirby and Paris also mention the Hydra battle, a game involving finite trees which is always won by Hercules, but not provably so in  $\text{PA}^2$ . Based on Kruskal’s tree theorem [28] Friedman introduced the following *miniaturisation*:

**4 Theorem** (Kruskal’s tree theorem). *For every  $l$  there exists a  $K$  such that for every sequence  $T_0, \dots, T_K$  of trees, where the  $T_i$  have at most  $l + i$  vertices, there exists  $i < j$  such that  $T_i$  is embeddable in  $T_j$ .*

This theorem is unprovable in PA (Friedman, around 1980). Recently Friedman [16] reached a new level of naturalness in unprovability in PA by showing that the same is true for the following theorem:

**5 Theorem** (finite adjacent Ramsey, AR). *For every  $k, r$  there exists  $R$  such that for every limited function  $C: R^k \rightarrow \mathbb{N}^r$  there are  $x_1 < \dots < x_{k+1} < R$  with  $C(x_1, \dots, x_k) \leq C(x_2, \dots, x_{k+1})$ .*

---

<sup>2</sup>These results on Goodstein sequences and Hydra battles can be traced back to Gentzen showing in 1936 that transfinite induction up to  $\varepsilon_0$  is not provable in PA formulated with a free predicate variable [18].

We will closely examine this theorem in Chapter 6. Other examples of unprovability involve concepts like arboreal numbers, kirality and  $\alpha$ -largeness.

Many unprovability results have been obtained for weaker theories. Interest in unprovability in those fragments can be motivated by the suspicion that for most natural finite theorems from the mathematics literature PA is overkill. In this context one should mention Friedman's grand conjecture:

Every theorem published in the Annals of Mathematics whose statement involves only finite mathematical objects (i.e., what logicians call an arithmetical statement) can be proved in EFA. EFA is the weak fragment of Peano Arithmetic based on the usual quantifier free axioms for  $0, 1, +, \times, \exp$ , together with the scheme of induction for all formulas in the language all of whose quantifiers are bounded. This has not even been carefully established for Peano Arithmetic. It is widely believed to be true for Peano Arithmetic, and I think that in every case where a logician has taken the time to learn the proofs, that logician also sees how to prove the theorem in Peano Arithmetic. However, there are some proofs which are very difficult to understand for all but a few people that have appeared in the Annals of Mathematics - e.g., Wiles' proof of FLT.<sup>3</sup>

For a gentle introduction into showing unprovability the reader is referred to [33], [5] for proofs involving models, and to [9], [31] for proofs that use proof-theoretic results. An extensive overview of unprovability results can be found in the introduction of [17]. These make use of results and techniques from model theory [33], [22], recursion theory [40] and proof theory [7], [38], [2], [6].

Though there are many interesting unprovability results in theories stronger than Peano Arithmetic, like the independence of the continuum hypothesis or Friedman's exotic case in [17], these fall outside the scope of this thesis.

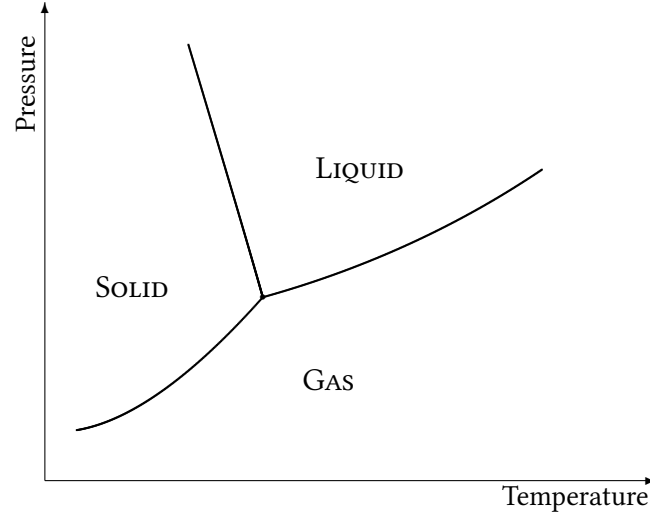
## 1.2 Phase transitions

During the developments in unprovability in the 1980's some interesting modifications of PA-independent statements emerged.

**6 Theorem** (MK<sub>r</sub>). *For every  $l$  there exists a  $K$  such that for every sequence  $T_0, \dots, T_K$  of trees, where the  $T_i$  have at most  $l + r \cdot \log(i)$  vertices, there exists  $i < j$  such that  $T_i$  is embeddable in  $T_j$ .*

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<sup>3</sup>Harvey Friedman, FOM posting: Grand conjectures, 16 April 1999 [14]



*Phase transitions in physics*

Loebl and Matoušek [30] showed that this variation is not provable in PA for  $r = 4$ , but is provable for  $r = \frac{1}{2}$ . The obvious question arising from this result is at which real  $c$  between those two the change from provability to provability occurs. Weiermann [44] provided the remarkable answer to this question, namely that it does so at  $c = \frac{1}{\log(\alpha)}$ , where  $\alpha = 2.9557652865 \dots$  is Otter's tree constant. For the Paris–Harrington and Kanamori–McAloon theorems the following versions were studied in [23]:

**7 Theorem** ( $\text{PH}_f$ ). *For every  $d, c, m$  there exists an  $R$  such that for every colouring  $C: [m, R]^d \rightarrow c$  there exists an  $H \subseteq [m, R]$  of size  $f(\min H)$  for which  $C$  limited to  $[H]^d$  is constant.*

**8 Theorem** ( $\text{KM}_f$ ). *For every  $d, m$  there exists  $R$  such that for every colouring  $C: [R]^d \rightarrow \mathbb{N}$  with  $C(x) \leq f(\min x)$  there exists  $H \subseteq R$  of size  $m$  for which for all  $x, y \in [H]^d$  with  $\min x = \min y$  we have  $C(x) = C(y)$ .*

In these cases the theorems become provable when  $f$  is a constant function, whilst  $f = \text{id}$  delivers the original statements. Similarly to the case of Kruskal's tree theorem, but with the parameter ranging over functions instead of rationals the obvious problem is one of classifying the parameter values according to the provability of the resulting theorem. For  $\text{PH}_f$  this was solved in [45] and for  $\text{KM}_f$  in [29]: Both variations turn out to be unprovable for  $\log^n$ , but provable for  $\log^*$ , the inverse of the tower function. These results have been refined and the variants with fixed dimension  $d$  examined in [29], [46] and [8].

A general programme was started by Weiermann to examine these kinds of phase transitions: Given a theorem  $A_f$  with parameter  $f$  that is not provable in a theory of arithmetic  $T$  for certain values of  $f$  and provable in others, determine the threshold at which the change from provability to unprovability occurs. A wealth of transition results has been obtained including for Goodstein sequences, Hydra Battles, Braid groups, Dickson's lemma, the Ackermann function, Higman's lemma and  $\alpha$ -largeness. An overview of these results can be found on Weiermann's webpage [43] and dissertations of Gyesik Lee [29] and Michiel De Smet [11].

## 1.3 Overview

We start with some preliminaries, assuming familiarity with (primitive) recursive functions, ordinals, basic logic, and results from proof theory. References for these are given here and the reader is advised to at least familiarise himself with ordinals below  $\varepsilon_0$  and recursive hierarchies. The reader should pay particular attention to the *How to prove it* sections and the comments on phase transitions at the end of the preliminaries.

In Chapters 2 and 3 we look at two simple examples of transition results related to  $n$ -tuples and König's lemma. These results involve theorems which are unprovable in  $\text{I}\Sigma_1$ , the weakest theory for which we will treat unprovable theorems. The reader who is unfamiliar with phase transitions is advised to start with those two examples. Additionally Chapter 2 makes for a great warming-up for Chapter 4. In this chapter we treat Maclagan's theorem, including a much needed cleaning of a published proof of unprovability from [37].

Chapters 5, 6 and 7 contain treatments of Ramsey-like unprovable theorems. Where in Chapter 7 we examine existing results using ideas from the treatment of the adjacent Ramsey theorem from Chapter 6, providing a unified treatment for these results. The proofs in Chapter 7 also offer some relatively simple proofs of transitions for the Ramsey-like theorems.

In Chapter 8 we discuss the possibility of generalising phase transitions. We suggest some tentative conjectures which provide a framework for the treatment of all transition results. In the second half of the chapter we also show some easy lemmas which describe the general proof-method of the provability parts of transitions. The last chapter contains a Dutch summary.

## 1.4 Preliminaries

### 1.4.1 Notations

We identify each natural number with the set of its predecessors:

$$R = \{0, \dots, R - 1\}.$$

**9 Definition.** For  $n$ -tuples  $x$  and  $y$  we say  $x \leq y$  if

$$(x)_1 \leq (y)_1 \wedge \dots \wedge (x)_n \leq (y)_n,$$

where  $(z)_i$  denotes the  $i$ 'th component of an  $n$ -tuple  $z$ .

**10 Definition.** We denote the set of  $d$ -element subsets of  $X$  with  $[X]^d$ . We use  $[n, R]^d$  instead of  $[\{n, \dots, R\}]^d$ . We call a function on such sets a colouring.

All functions will involve only natural numbers. If notation suggests a function  $f: \mathbb{N} \rightarrow \mathbb{R}$  we interpret it as the floor of  $f$ , written  $\lfloor f \rfloor$ . We use  $\log$  to denote the binary logarithm, where  $\log(0) = 0$ . Iterations of the function  $f$  are denoted by  $f^n$  as opposed to  $f(i)^n$  for exponentials and  $f^0$  is the identity function.

**11 Definition.** Given unbounded function  $f: \mathbb{N} \rightarrow \mathbb{N}$ , its inverse is:

$$f^{-1}(i) = \max\{j : f(j) \leq i\}.$$

**12 Definition.** The tower function with base 2 and height  $k$  is defined as follows:

$$\begin{aligned} 2_0(c) &= c, \\ 2_{k+1}(c) &= 2^{2_k(c)}. \end{aligned}$$

**13 Definition.** The function  $\log^*$  is the inverse of  $i \mapsto 2_i(2)$ .

**14 Definition.**  $i \% c = i \pmod{c}$ .

### 1.4.2 Ordinals

Ordinals extend the natural numbers similarly to cardinalities with the difference that not only size but also an order structure is taken into account. They are a tool for measuring the strength of a theory, where the stronger theories get associated with higher ordinals. The assignment of the smallest ordinal which a theory can

prove to be well-ordered, the proof theoretic ordinal, is studied in ordinal analysis (See, for example [3]). We will be working with fragments of PA whose associated ordinals are those up to  $\varepsilon_0$ . Intuitively these ordinals are, in increasing order:

$$0, 1, 2, 3, 4, 5, 6, \dots, \omega, \omega + 1, \omega + 2, \omega + 3, \dots,$$

$$\omega + \omega = \omega \cdot 2, \dots, \omega \cdot \omega = \omega^2, \dots, \omega^\omega = \omega_2, \dots, \omega^{\omega_2} = \omega_3, \dots, \omega_\omega = \varepsilon_0.$$

These are well-ordered, ordinals  $\alpha + 1$  are called successors, the remaining non-zero ordinals limits. As implied, addition, multiplication and exponentiation on the ordinals are defined using transfinite recursion:

**15 Definition.**

1.  $\alpha + 0 = \alpha$ ,
2.  $\alpha + (\beta + 1) = (\alpha + \beta) + 1$ ,
3.  $\alpha + \gamma = \sup_{\beta < \gamma} (\alpha + \beta)$ , where  $\gamma$  is a limit.

**16 Definition.**

1.  $\alpha \cdot 0 = 0$ ,
2.  $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$ ,
3.  $\alpha \cdot \gamma = \sup_{\beta < \gamma} (\alpha \cdot \beta)$ , where  $\gamma$  is a limit.

**17 Definition.**

1.  $\alpha^0 = 1$ ,
2.  $\alpha^{(\beta+1)} = (\alpha^\beta) \cdot \alpha$ ,
3.  $\alpha^\gamma = \sup_{0 < \beta < \gamma} (\alpha^\beta)$ , where  $\gamma$  is a limit.

Even though these functions have a similarity with arithmetic on the natural numbers, there are differences. For example addition is not commutative.

**18 Definition.**  $\omega_0(l) = l$ ,  $\omega_{n+1}(l) = \omega^{\omega_n(l)}$  and  $\omega_n = \omega_n(1)$ .

**19 Definition.** All  $\alpha < \varepsilon_0$  can be written uniquely in the Cantor Normal Form (CNF):

$$\alpha = \omega^{\alpha_1} \cdot m_1 + \dots + \omega^{\alpha_n} \cdot m_n,$$

where  $\alpha_1 > \dots > \alpha_n$  and  $m_1 > 0, \dots, m_n > 0, n \geq 1$ . The CNF of 0 is  $\omega^0 \cdot 0$ .

We use the canonical fundamental sequences:

**20 Definition.**

$$\begin{aligned}
(\alpha + 1)[x] &= \alpha, \\
(\alpha + \omega^{\alpha_n+1} \cdot (m + 1))[x] &= \alpha + \omega^{\alpha_n+1} \cdot m + \omega^{\alpha_n} \cdot x, \\
(\alpha + \omega^\gamma \cdot (m + 1))[x] &= \alpha + \omega^\gamma \cdot m + \omega^{\gamma[x]},
\end{aligned}$$

where  $\gamma$  is a limit, the ordinals are  $< \varepsilon_0$  and in CNF. Furthermore  $\varepsilon_0[x] = \omega_x$  and  $0[x] = 0$ .

Fundamental sequences for limits are sequences of ordinals with which one approximates those limits. They allow us to use ordinals as a bookkeeping device for increasing ‘the amount of diagonalisation’ in the recursive hierarchies. From the viewpoint of provability it is important that these sequences should not be increasing ‘too slowly’, but our choice here is well within safety limits. More details on ordinals can be found in [7], [38], [2] and [6].

**1.4.3 Primitive recursive functions**

**21 Definition.** *The set of primitive recursive functions is the smallest set such that:*

1. *It contains the constant functions, successor function and the projection functions,*
2. *it is closed under composition and*
3. *it is closed under the scheme of primitive recursion: If  $g$  and  $h$  are primitive recursive, then so is (assuming proper arities) the following  $f$ :*

$$\begin{aligned}
f(0, \bar{x}) &= g(\bar{x}), \\
f(n + 1, \bar{x}) &= h(n, \bar{x}, f(n, \bar{x})).
\end{aligned}$$

**22 Definition.** *The set of functions that are primitive recursive in a function  $f$  is defined as the set of primitive recursive functions, where at item 1 of the definition  $f$  is added.*

The primitive recursive functions include many often used functions like, for example, addition, multiplication, exponentiation, tower functions, and Ramsey numbers. The scheme of primitive recursion also allows for the bounded search operator, though it does not allow unbounded search. If one adds unbounded search as a closure property we obtain the recursive or computable functions.



### 1.4.4 Recursive Hierarchies

We will be using the recursive hierarchies to provide lower bound estimates on existential witnesses of theorems. This is an important step in proving independence because recursive hierarchies provide upper bounds for the witnesses of provable theorems.

**23 Definition.** *The Ackermann hierarchy is:*

$$\begin{aligned} A_0(i) &= i + 1, \\ A_{n+1}(i) &= A_n^i(i), \\ A(i) &= A_i(i). \end{aligned}$$

*We call  $A$  the Ackermann function and any function that eventually dominates every  $A_n$  Ackermannian.*

The Ackermann function is a modified version of the function introduced by Ackermann [1] to show that not all computable functions are primitive recursive. Like this function, the Ackermann function is Ackermannian. One shows this by verifying that every primitive recursive function can be eventually bounded by an  $A_n$ .

**24 Definition.** *The fast growing hierarchy is:*

$$\begin{aligned} F_0(i) &= i + 1, \\ F_{\alpha+1}(i) &= F_\alpha^i(i), \\ F_\gamma(i) &= F_{\gamma[i]}(i). \end{aligned}$$

Note that  $A = F_\omega$ .

**25 Definition.** *The Hardy hierarchy is:*

$$\begin{aligned} H_0(i) &= i, \\ H_{\alpha+1}(i) &= H_\alpha(i + 1), \\ H_\gamma(i) &= H_{\gamma[i]}(i + 1). \end{aligned}$$

In these hierarchies functions at level  $\omega_n$  ( $n > 1$ ) are not primitive recursive in functions at level  $\alpha < \omega_n$ . More details on these hierarchies can be found in [40].

### 1.4.5 Theories of Arithmetic

7.  $a, b \in \mathbb{N} \rightarrow a = b \Rightarrow a + 1 = b + 1.$
8.  $a \in \mathbb{N} \rightarrow a + 1 \neq 0.$
9.  $k \in \mathbb{K} \rightarrow 1 \in k \rightarrow x \in \mathbb{N} \rightarrow x \in k \rightarrow \mathcal{O}_x. x + 1 \in k : \mathcal{O}. \mathbb{N} \subset k.$
18.  $a, b \in \mathbb{N} \rightarrow a + (b + 1) = (a + b) + 1.$
1.  $a \in \mathbb{N} \rightarrow a \times 1 = a.$
2.  $a, b \in \mathbb{N} \rightarrow a \times (b + 1) = a \times b + a.$

Giuseppe Peano, 1889

Based on and named after the principles of arithmetic which Giuseppe Peano introduced in [36], Peano Arithmetic is capable of proving almost all of natural finite mathematics. Its main features are the simple rules of arithmetic and the principle of induction on formulas from the language of arithmetic. By limiting this axiom scheme one obtains fragments of the theory which by themselves have interesting unprovable statements. Often one can get theorems which are provable in Peano Arithmetic but not in several of those fragments by weakening the theorems which are unprovable in PA, examples of this would be PH and KM where this can be done by just fixing the dimensions.

**26 Definition.** Take language  $\mathcal{L} = \{+, \times, <, 0, 1\}$ . The first order theory of Peano Arithmetic (PA) consists of:

1. Distributivity, associativity and commutativity of  $+$  and  $\times$  with neutral elements 0 and 1 respectively.
2.  $<$  is a discrete linear order, with minimal element 0, 1 is the successor of 0.
3.  $x < y \rightarrow x + z < y + z.$

With the axiom scheme for induction: For every  $\mathcal{L}$ -formula  $\varphi$  the universal closure of:

$$[\varphi(0, \bar{y}) \wedge \forall x(\varphi(x, \bar{y}) \rightarrow \varphi(x + 1, \bar{y}))] \rightarrow \forall x \varphi(x, \bar{y}).$$

**27 Definition.**

1.  $\Delta_0$ -,  $\Sigma_0$ - and  $\Pi_0$ -formulas are formulas whose quantifiers are bounded.
2.  $\Sigma_{n+1}$ -formulas are formulas which have the shape  $\exists x_1 \dots x_i \varphi$ , where  $\varphi$  is a  $\Pi_n$ -formula.
3.  $\Pi_{n+1}$ -formulas are formulas which have the shape  $\forall x_1 \dots x_j \varphi$ , where  $\varphi$  is a  $\Sigma_n$ -formula.

So  $\Sigma_n$ -formulas have the following shape:

$$\exists x_1 \dots x_{i_1} \forall y_1 \dots y_{i_2} \exists \dots \varphi,$$

where there are  $n$  quantifiers and  $\varphi$  contains only bounded quantifiers. All our results will involve theories in which encodings of tuples are no problem, hence we will assume without loss of generality that the  $i_1, \dots, i_n$  are all equal to 1. Information on encoding tuples can be found in, for example, [7].

**28 Definition.** *The theory  $I\Sigma_n$  is PA with the induction axioms restricted to  $\Sigma_n$ -formulas.*

**29 Definition.** *Given a theory  $T$ , we call a function  $f: \mathbb{N} \rightarrow \mathbb{N}$  provably total or provably recursive in  $T$  if there exists a  $\Sigma_1$ -formula  $\varphi$  such that:*

1.  $f(x) = y$  if and only if  $\varphi(x, y)$  holds and
2.  $T \vdash \forall x \exists! y \varphi(x, y)$ .

Note that point 1 in this definition is equivalent to  $f$  being computable. We will often omit  $T$  when it is clear from the context in which theory we are examining (un)provability.

### 1.4.6 How to prove it: Showing unprovability

We make use of the following connections between function hierarchies and provably total functions. More details on these results from proof theory can be found in [6], [2], [38] and [7].

**30 Theorem.** *A computable function is provably total in PA if and only if it is primitive recursive in  $F_\alpha$  for an  $\alpha < \varepsilon_0$ .*

**31 Theorem.** *For  $n > 0$  a computable function is provably total in  $I\Sigma_n$  if and only if it is primitive recursive in  $H_\alpha$  for an  $\alpha < \omega_{n+1}$ .*

**32 Theorem.** *For  $n > 0$  a computable function is provably total in  $I\Sigma_n$  if and only if it is primitive recursive in  $F_\alpha$  for an  $\alpha < \omega_n$ .*

All theorems we examine will have the shape  $\forall x \exists y \varphi(x, y)$  with  $\varphi$  being a  $\Delta_0$ -formula. The provability of such a theorem is equivalent to the function  $M: x \mapsto \min\{y : \varphi(x, y)\}$  being provably total. In other words: If, using the closure properties of the primitive recursive functions, one obtains from  $M$  a function which

eventually dominates all elements of the hierarchy involved then the original theorem is not provable. We will be using a much weaker version of this principle which involves, intuitively speaking:

$$\begin{array}{ccccc} M & \circ & h & \geq & H, \\ \uparrow & & \uparrow & & \uparrow \\ \text{fast} & \Leftarrow & \text{slow} & \& & \text{fast} \end{array}$$

where fast and slow are relative to the theory in which we are showing unprovability. The term fast (growing) will indicate that the function eventually dominates every element of the hierarchy involved, slow (growing) indicates it is eventually dominated by an element of this hierarchy.

**In these proofs the precise shape of the function  $h$  is not important!** We use only that this function is ‘fast enough for the proof to work’, though still slow. For understanding the proofs in this thesis it is essential to keep this in mind.

A classic method of showing unprovability is to construct a model of the theory in which the theorem is not true. This often involves obtaining a sequence of elements, called indiscernibles, in a nonstandard model. The existence of indiscernibles is derived from the theorem being true in the standard part of the model. The properties of the indiscernibles allow one to take an initial segment of the non-standard model in which the unprovable theorem is not true.

#### 1.4.7 How to prove it: Showing provability

The most obvious way to show that a statement is provable in a theory  $T$  is to prove the theorem in  $T$ . We will not use such a method but use properties derived from theorems 30, 31, 32 and the fact that the scheme of primitive recursion allows for bounded search. We start with a theorem  $\forall x \exists y \varphi$  with  $\varphi$  a  $\Delta_0$ -formula. If an existential witness to this theorem can be bounded by a function  $h$  that is primitive recursive in the appropriate hierarchy then  $x \mapsto \min\{y < h(x) : \varphi(x, y)\}$  is also primitive recursive in that hierarchy, hence provably total in the associated theory, showing the theorem itself is also provable.

#### 1.4.8 Phase transitions

It is often possible to point at a certain place in an unprovable arithmetical theorem  $\varphi$  which appears to make the theorem unprovable. For example in PH this

is the size of the set  $H$ , in KM the bounds on the range, in Kruskal's theorem the bound on the complexity of the trees. At this point replacing this bound with a fixed value  $c$  delivers a provable theorem  $\varphi_c$ , often proved using ordinary Ramsey theory or some counting plus the pigeonhole principle. At this position we introduce the parameter function to obtain  $\varphi_f$ , the parametrised version of the theorem. This parametrised version of  $\varphi$  is provable for any constant function (the weak versions:  $\varphi_c$ ), but not provable for the identity (the strong version:  $\varphi$ ). Classifying functions  $f$  according to the provability of the resulting theorem  $\varphi_f$  becomes a natural problem. We will always assume the parameter functions to be computable and nondecreasing.

The resulting classification seems to depend on the combinatorics of the provable versions of the theorem under examination. This indicates a correspondence between mathematics happening at and below the level of the tower function and provability of the parametrised version of the theorem. For example  $\text{PH}_f$  (or  $\text{KM}_f$  or  $\text{AR}_f$ ) is unprovable for  $\log^n$ , whilst iterated exponential lower bounds for Ramsey numbers can be obtained for sufficiently high dimension. Ramsey numbers can be eventually bounded by the tower function, this has as a result that  $\text{PH}_f$  becomes provable for  $\log^*$ . This is also seen in fragments of PA, where for theorems not provable in  $\text{I}\Sigma_{n+1}$  the corresponding parametrised theorem is provable for  $\log^{n+1}$  but not provable for  $\log^n$ . Similarly transition results for Kruskal's theorem, Maclagan's theorem, sequences of ordinals, braid groups depend on solutions to counting problems of those objects, resulting in questions of interest to the fields to which the theorems are related to.

This connection is clearly seen in Chapter 6: Adjacent Ramsey. The estimates for 'provable' adjacent Ramsey numbers are directly plugged into numbers which are not witnesses to the theorem. The freedom to select the dimension in the range of the colourings allows for extra coordinates to be used for this plugging. In this context we would also like to highlight Chapter 7. Though the transition results in that chapter have been shown previously, the proofs in that chapter directly demonstrate the influence of the finite combinatorics on the classification of the parameter values. They also clarify the difference between the transitions of the Paris–Harrington and the Kanamori–McAloon theorems. Interestingly this is done using the same compression technique for all three of those theorems.

It should be noted that one does not always need such lower bound results to obtain the unprovable part of the transition. In the second part of Chapter 5 we will demonstrate a treatment of a variation on the Kanamori–McAloon theorem where

all the combinatorial problems will be treated by using nonstandard models. The two main principles in this proof involve the construction of a model as in [23] whilst viewing certain numbers as encodings of finite sets. The downside is that we will be using extra dimensions to make this possible, making it impossible to use this technique for fragments of PA and limited dimensions.

The sharpening of the thresholds also follows certain heuristics. In the Ramsey type examples we see that unprovability for all  $\log^n$  results in unprovability for  $i \mapsto \log^{H_{\varepsilon_0}^{-1}(i)}(i)$ , whilst the provability argument for  $\log^*$  works just as well for  $i \mapsto \log^{H_{\alpha}^{-1}(i)}(i)$  with  $\alpha < \varepsilon_0$ .

For fragments of PA we see something similar: If, in  $\text{I}\Sigma_{n+1}$ ,  $\varphi_f$  is unprovable for  $\sqrt[n]{\log^n}$ , then the theorem with parameter  $i \mapsto {}^{H_{\omega_{n+2}}^{-1}(i)}\sqrt[n]{\log^n i}$  is not provable in  $\text{I}\Sigma_n$  where it becomes provable when  $\omega_{n+2}$  is replaced with a lower ordinal. In the case that we have unprovability for  $\frac{\log^{n+1}}{c}$  we have that the theorem is not provable for  $i \mapsto \frac{\log^n}{H_{\omega_{n+2}}^{-1}(i)}$ .

There appears to be a connection with the fact that, even though  $H_{\varepsilon_0}^{-1}$  is provably total in EFA, it is not provably unbounded in PA. We suspect that it is possible to use this to provide a general proof, involving nonstandard models, of a theorem that unprovability of  $\varphi_{f_c}$  in PA implies unprovability of  $\varphi_f$ , where  $f(i) = f_{H_{\varepsilon_0}^{-1}(i)}(i)$ . If this is possible we expect this to generalise to the fragments of PA with ease. At this moment the different sharpenings of transition results have involved ad-hoc arguments. If we use recursion theoretic estimates for proving independence we use those for the sharpening. If we use a model theoretic approach we split the argument into two cases, where in one case  $\varphi_f$  is unprovable by a lower bound estimate and in the other case we provide the model construction for a nonstandard  $c$ .

We will give a more thorough examination of these observations in Chapter 8.

## Chapter 2

# Sequences of tuples

This chapter is related to a famous result: The Friedman style miniaturisation of Dickson's lemma. Dickson's lemma itself has been attributed to L. E. Dickson [12]. The principle MH and its unprovability are attributed to Friedman in [34].

**33 Lemma (Dickson).** *Every sequence of  $k$ -tuples  $m_0, m_1, \dots$  such that  $m_i \not\leq m_j$  for all  $i < j$  is finite.*

For the miniaturisation we need a complexity measure of tuples:

**34 Definition.** *The degree of a  $k$ -tuple is the sum of its coordinates:  $\deg(m) = (m)_1 + \dots + (m)_k$ .*

**35 Lemma (MDL).** *For all  $k, l$  there exists  $M$  such that for every sequence  $m_0, \dots, m_M$  of  $k$ -tuples with  $\deg(m_i) \leq l + i$  for all  $i \leq M$  there exist  $i < j \leq M$  such that  $m_i \leq m_j$ .*

Dickson's lemma is well studied (showing that this finite version is unprovable in  $\text{IS}_1$  is an excellent exercise in an introductory course on proof theory) and a phase transition has been determined. We examine the following version (using  $g_i(x)$  instead of  $(g(x))_i$ ):

**36 Lemma (MH).** *For all  $k, m$  there exists  $H$  such that for all  $g: H \rightarrow \mathbb{N}^k$  with  $g_i(x) \leq x$  for all  $i \leq k, x < H$  there exists sequence  $b_1 < \dots < b_m < H$  such that for all  $i \leq k$  either  $\forall j < m, b_j < g_i(b_{j+1})$  or  $g_i$  is constant on the  $b$ 's.*

This lemma asserts the existence of a subsequence of increasing tuples of length  $m$  in every sequence of tuples for which the coordinates are linearly bounded. Furthermore, the different coordinates of those tuples will be either constant for the entire subsequence or will increase with a growth rate depending on the position in this subsequence. Compared to MDL some naturalness has been sacrificed to obtain a statement which is more surprising.

MH can be shown by proving an infinite version of this lemma using infinite Ramsey for pairs first, obtaining the finite version with compactness.

**37 Definition.** *Denote the least such  $H$  with  $\text{MH}^k(m)$ .*

**38 Theorem.**  $\text{IS}_1 \not\vdash \text{MH}$ .

*Proof:* Take  $M^k(1)$  equal to the least  $M$  from lemma 35 with  $m = 1$ . We show  $MH^k(3) \geq M^k(1)$ . Given  $m_0, \dots, m_H$  with  $\deg(m_j) \leq 1 + j$ , define  $g(0) = 0$  and  $g(x) = m_{x-1}$  for  $0 < x < H$ . Take  $b_1 < b_2 < b_3$  such that for all  $i \leq k$  either  $\forall j \leq 3 \ b_j < g_i(b_{j+1})$  or  $g_i(b_1) = g_i(b_2) = g_i(b_3)$ . Notice that in the first case:

$$0 \leq b_1 < g_i(b_2) \leq 1 + b_2 - 1 < g_i(b_3).$$

Hence  $m_{b_2-1} \leq m_{b_3-1}$ .

□

**39 Lemma ( $MH_f$ ).** *For all  $k, m$  there exists  $H$  such that for all  $g: H \rightarrow \mathbb{N}^k$ , with  $g_i(x) \leq f(x)$  for all  $i \leq k, x < H$ , there exists sequence  $b_1 < \dots < b_m < H$  such that for all  $i \leq k$  either  $\forall j < m \ b_j < g_i(b_{j+1})$  or  $g_i$  is constant on the  $b$ 's.*

**40 Definition.** Denote the least such  $H$  with  $MH_f^k(m)$ .

**41 Theorem.**  $\text{IS}_1 \not\vdash MH_{\sqrt[\mathbb{Q}]{\cdot}}$ , but  $\text{IS}_1 \vdash MH_{\log}$ .

*Proof:* We show that  $MH_{\sqrt[\mathbb{Q}]{\cdot}}^{k+c+1} \geq MH_{\text{id}}^k$ . First note that the number of different  $(c+1)$ -tuples with coordinates  $\leq a$  exceeds  $(a+1)^c$ , we denote one sequence enumerating such tuples with  $m_0(a), \dots, m_{(a+1)^c}(a)$ . Given bad function  $g: R \rightarrow \mathbb{N}^k$  for  $\text{id}$ , define bad function  $h$  for  $\sqrt[\mathbb{Q}]{\cdot}$ :

$$h(x) = (g(\sqrt[c]{x}), m_{x - \sqrt[c]{x}^c}(\sqrt[c]{x})).$$

For the provability part notice that, if one has  $m \cdot (c+1)^k$   $k$ -tuples with coordinates  $\leq c$ , then at least  $m$  of them are identical. So  $MH_{\log}^k(m) \leq 2^{mk}$  for  $m$  sufficiently large.

□

**42 Theorem.** Let  $f(i) = {}^{A^{-1}(i)}\sqrt{i}$  and  $f_n(i) = {}^{A_n^{-1}(i)}\sqrt{i}$ , then:  
 $\text{IS}_1 \not\vdash MH_f$ , but  $\text{IS}_1 \vdash MH_{f_n}$ .

*Proof:* For this we need to actually provide lower bounds.  $M_{\text{id}}^{2k+1}(l) \geq A_k(l)$  can be shown by constructing bad sequences of appropriate length using recursion on  $k$ . Furthermore  $M^{k+l}(1) \geq M^k(l)$ . Therefore  $MH_{\sqrt[\mathbb{Q}]{\cdot}}^{3k+c+2}(3) \geq A_k(k)$ . We claim that:

$$H = MH_f^{4k+3}(3) \geq A(k).$$

Assume for contradiction that  $H < A_k(k)$ , then for  $x \leq H$  we know  $A^{-1}(x) \leq k$ , hence  $f(x) \geq \sqrt[k]{x}$ . So:

$$\begin{aligned} H &\geq MH_{\sqrt[\mathbb{Q}]{\cdot}}^{4k+3}(3) \\ &\geq A_k(k+1), \end{aligned}$$



which contradicts our assumption.

For the provability part we claim that  $\text{MH}_{f_n}^k(m) \leq m^{4A_n(4k+4)} = M$ . If  $x \leq M$  then  $f_n(x) \leq f_n(M) \leq \sqrt[4k+4]{m^{4A_n(4k+4)}}$ . Hence by the estimate from the provability part of the previous theorem the number of different  $k$ -tuples required to have  $m$  identical elements has upper bound  $(m^{\frac{A_n(4k+4)}{k+1}} + 1)^{k+1} < m^{4A_n(4k+4)}$ .

□

## 2.1 Dickson's lemma

For completeness we include a result from Weiermann on Dickson's lemma. We see here already the general shape of the proof of the transition result in chapter 4.

**43 Lemma** ( $\text{MDL}_f$ ). *For all  $k$  and  $l$  there exists  $M$  such that for every sequence  $m_0, \dots, m_M$  of  $k$ -tuples with  $\deg(m_i) \leq l + f(i)$  for all  $i \leq M$  there exist  $i < j \leq M$  such that  $m_i \leq m_j$ .*

**44 Definition.** We denote the least  $M$  from  $\text{MDL}_f$  with  $D_k^f(l)$ .

**45 Lemma.** *If  $f(i) = \sqrt[c]{i}$  then  $\text{IS}_1 \not\vdash \text{MDL}_f$ .*

We first show  $D_{d+c}^{(c+1) \cdot f}(l+c) \geq D_d^{id}(l)$ :

Note that  $D_c^0(l+c) > (l+1)^c$  (induction on  $c$ ), denote the sequences that show this with  $z(l)$ , take sequence  $b$  in  $d$  variables. Construct:

$$m_i = (z_{i-f(i)^c}(f(i)), b_{f(i)}).$$

We show:  $D_{d+1}^h(l+c) \geq D_d^g(l)$  for any  $h(i) = g(\frac{i}{c})$ :

Take sequence  $b$  in  $d$  variables, construct:

$$m_i = (c - i \% c, b_{\frac{i}{c}}).$$

□

**46 Theorem** (Weiermann). *If  $f(i) = \sqrt[A^{-1}(i)]{i}$  then  $\text{IS}_1 \not\vdash \text{MDL}_f$ .*

*Proof:* Take  $f_c(i) = \sqrt[c]{i}$ . We show:  $D_{3l+2}^f(2l + (l+1)^l + 1) > A(l)$ . Assume, for a contradiction, that  $N(l) = D_{3l+2}^f(2l + (l+1)^l + 1) \leq A(l)$ . Then  $A^{-1}(i) \leq l$  for

any  $i \leq N(l)$ , hence  $\sqrt[l]{i} \leq {}^{A^{-1}(i)}\sqrt{i}$  for such  $i$ . So:

$$\begin{aligned} N(l) &\geq D_{3l+2}^{f_l}(2l+1+(l+1)^l) \\ &\geq D_{3l+1}^{(l+1)f_l}(2l+1) \\ &\geq D_{2l+1}^{ld}(l+1) > A_l(l) = A(l), \end{aligned}$$

in contradiction with our assumption. □

**47 Theorem (Weiermann).** *If  $B$  is primitive recursive, increasing, unbounded and  $f(i) = {}^{B^{-1}(i)}\sqrt{i}$  then  $\text{IS}_1 \vdash \text{MDL}_f$ .*

*Proof:* Assume without loss of generality that  $B(l) > l^d$ . We show that  $M_d^f(l) \leq B(l)^d$  for  $l \geq 4d$ . Observe first that  $N(l) = \#\{m \in \mathbb{N}^d : \deg(m) \leq l\} \leq l^d$ . Hence any sequence of length  $B(l)^d + 1$  contains at most:

$$\begin{aligned} N(l + {}^{B^{-1}(l)}\sqrt{B(l)^d}) &\leq (l + \sqrt[l]{B(l)^d})^d \\ &\leq (2\sqrt[l]{B(l)^d})^d < B(l)^d + 1 \end{aligned}$$

different  $d$ -tuples. Hence by pigeonhole principle any such sequence must contain two identical elements. □

On a small side note, examine the following variant:

**48 Lemma ( $\text{MDL}_f^r$ ).** *For all  $k, l, r > 0$  there exists  $M$  such that for every sequence  $m_0, \dots, m_M$  of  $k$ -tuples with  $\deg(m_i) \leq l + f(i)$  for all  $i \leq M$  there exist  $i_0 < \dots < i_r \leq M$  such that  $m_{i_0} \leq m_{i_1} \leq \dots \leq m_{i_r}$ .*

This lemma has identical transition result to MDL. The reason for this is as follows:

For  $r = 1$  we have already unprovability for  $f(i) = \sqrt[i]{i}$  and for  $f(i) = {}^{A^{-1}(i)}\sqrt{i}$ , hence also for  $r > 1$ . For the provability part suppose  $B(l) > l^{k+1}$  is increasing and primitive recursive. Examine the estimate:  $N(l) = \#\{m \in \mathbb{N}^k : \deg(m) \leq l\} \leq l^k$ . Then any sequence of length  $r \cdot B(l)^k + 1$  contains at most:

$$\begin{aligned} N(l + {}^{B^{-1}(l)}\sqrt{r \cdot B(l)^k}) &\leq (l + \sqrt[l]{r \cdot B(l)^k})^k \\ &\leq (2\sqrt[l]{r \cdot B(l)^k})^k < B(l)^k + 1 \end{aligned}$$

different  $d$ -tuples for sufficiently large  $l$ . Hence by pigeonhole principle any such sequence must contain  $r$  identical elements. □

## Chapter 3

# König's lemma

Named after Dénes Kőnig this lemma originally was part of graph theory [27]. It plays a very important role in logic. Many of the unprovable theorems are obtained from infinite versions using a compactness argument with this lemma and it plays an important part in reverse mathematics, as it is equivalent to  $\text{ACA}_0$  over  $\text{RCA}_0$  [41].

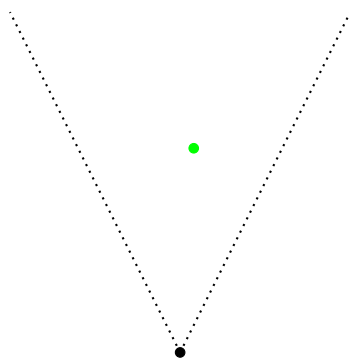
**49 Lemma** (König's Lemma). *Every finitely branching infinite tree contains an infinite path.*

The example from this chapter is closely related to  $c$ -arboreal sets in [34]. In particular a theorem similar to theorem 55 is proved there. In principle this is a miniaturisation of König's lemma.

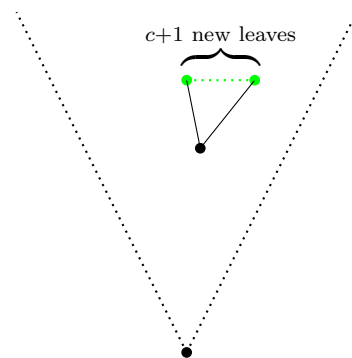
We begin with an innocent procedure:

**50 Definition** (Growing Tree <sub>$c$</sub> ). *Construct a finite tree as follows: Start with a root and at step  $i$  append  $c + 1$  leaves on one of the existing leaves.*

At each step there is a choice of a leaf, so a tree grown in a certain number of steps is not uniquely determined.



Step  $i$ : Select a leaf



Step  $i$ : Add new leaves

Because the number of leaves that is added in each step is constant we have the following:

**51 Lemma** ( $\text{MKL}_c$ ). *For every  $h$  there exists a  $K$  such that any Growing Tree <sub>$c$</sub>  grown in  $K$  steps has reached at least height  $h$ .*

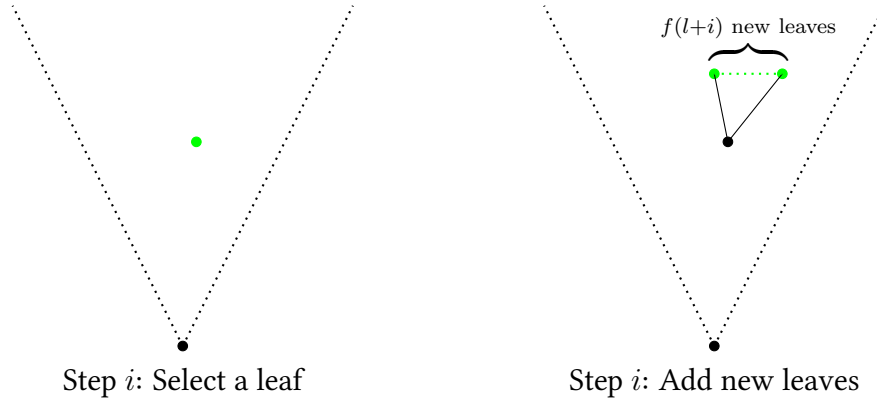
**52 Lemma.**  $\text{EFA} \vdash \text{MKL}_c$ .

*Proof:*  $(c + 1)^h$  steps are sufficient to grow a tree of height  $h$ .

□

We modify the growing of trees to produce something much less innocent.

**53 Definition** (Growing  $l$ -Tree <sub>$f$</sub> ). *Construct a finite tree as follows: Start with a root and at step  $i$  append  $f(l + i)$  leaves on one of the existing leaves.*



By König's lemma any growing tree will reach any height after a sufficient number of steps.

**54 Lemma** ( $\text{MKL}_f$ ). *For every  $h, l$  there exists a  $K$  such that any Growing  $l$ -Tree <sub>$f$</sub>  grown in  $K$  steps has reached at least height  $h$ .*

*Proof:* Apply König's lemma to König's lemma as follows: Given  $h, l$  we construct a labelled tree which has itself trees on the labels. For convenience we call this tree a meta-tree.

The root is labelled with the tree consisting of the root and  $f(l + j)$  leaves, where  $j$  is the least such that  $f(l + j) > 0$ .

At each node  $n$  of the meta-tree, if the tree  $T$  on the label of  $n$  has height  $h$ , then  $n$  is a leaf. If  $T$  does not have height  $h$  and it is a  $\text{Tree}_f$  grown in  $i$  steps then  $n$  has a direct descendant labelled with  $T'$  for each possible  $T'$  grown in one step from  $T$ .

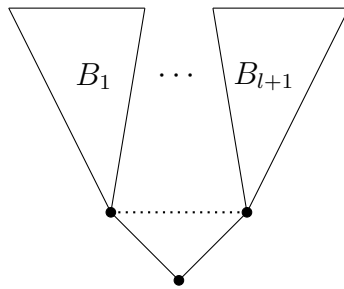
This is a finitely branching meta-tree, because at each step in growing trees there is only a finite number of possible choices of leaves. Assume for a contradiction that for all  $K$  there exists a Growing  $l$ - $\text{Tree}_f$ , grown in  $K$  steps, that has not reached height  $h$ . Then our meta-tree is infinite, hence by König's lemma it has an infinite path. Examine the tree obtained by taking the union of the trees that are labels on this meta-path. This tree has a height of at most  $h$  because it is the union of trees of such height. It is an infinite tree because at each step in growing trees new nodes are added. It is finitely branching because at each step in growing trees only a finite number of nodes is added and those nodes are only added to leaves. Hence by König's lemma this tree has an infinite path, which is in contradiction to the height of this tree being at most  $h$ .

□

For  $h, l, f$  we call a Growing  $l$ - $\text{Tree}_f$  *bad* if it has not reached height  $h$ . We define the function  $K_h^f(l)$  to be the maximum number of steps in which such a bad tree can be grown. This is properly defined thanks to  $\text{MKL}_f$ .

**55 Theorem.**  $\text{IS}_1 \not\models \text{MKL}_{id}$ .

*Proof:* First note that  $K_3^{id}(l) > l + 1$ . Furthermore  $K_{3+h}^{id}(l) > A_h(l)$  can then be shown with induction on  $h$ .



Construction of a bad tree for height  $h + 1 + 3$  using trees for height  $h + 3$ : Bad trees  $B_i$  are constructed in more than  $A_h^i(l)$  steps.

□

**56 Theorem.** If  $f(i) = \sqrt[i]{i}$  then  $\text{IS}_1 \not\models \text{MKL}_f$ .

*Proof:* First note that  $K_{c+3}^f(l) > l + 1$ . Define:

$$\begin{aligned} A_0^f(i) &= i + 1 \\ A_{n+1}^f(i) &= (A_n^f)^{f(i)}(i) \\ A^f(i) &= A_i^f(i) \end{aligned}$$

Observe that  $A_n(i) \leq A_{n+2c^2+1}^f(i)$  for any  $i \geq 4^c$  (See [8], corollary 3.3). Furthermore  $K_{c+3+h}^f(l) > A_h^f(l)$  can be shown again with induction on  $h$ .

□

**57 Theorem.** *If  $f(i) = {}^{A^{-1}(i)}\sqrt{i}$  then  $\text{IS}_1 \not\models \text{MKL}_f$ .*

*Proof:* Take  $f_c(i) = {}^c\sqrt{i}$ . We show  $N(l) = K_{2l^2+2l+4}^f(l) > A(l)$  for  $l \geq 4^c$ . Assume for a contradiction that  $N(l) \leq A(l)$ , thus  $A^{-1}(i) \leq l$  for any  $i \leq N(l)$ , hence  ${}^l\sqrt{i} \leq {}^{A^{-1}(i)}\sqrt{i}$  for such  $i$ . So:

$$\begin{aligned} N(l) &\geq K_{2l^2+2l+4}^{f_l}(l) \\ &> A_{2l^2+l+1}^l(l) \\ &\geq A_l(l) = A(l) \end{aligned}$$

in contradiction with our assumption.

□

**58 Theorem.** *If  $B$  is primitive recursive, increasing and unbounded and  $f(i) = {}^{B^{-1}(i)}\sqrt{i}$  then  $\text{IS}_1 \vdash \text{MKL}_f$ .*

*Proof:* Assume without loss of generality that  $B(l) > l^h$ . We claim that  $K_h^f(l) \leq B(l)^h$  for  $l > 3h$ . Suppose we have a construction of a tree in  $B(l)^h$  steps. Then in each step we have added at most  ${}^{B^{-1}(l+B(l)^h)}\sqrt{l+B(l)^h} \leq (2B(l))^{\frac{h}{t}}$  leaves. But the number of steps required to reach height  $h$  in the construction of such a tree is less than  $((2B(l))^{\frac{h}{t}})^h < B(l)^h$ .

□

The process of growing trees can be simplified whilst keeping the same transition results:

**59 Definition.** *Construct a finite tree as follows: Start with a root and at step  $i$  append  $f(i)$  leaves on one of the existing leaves.*

Here we retain the same growing trees, but they start after a polynomial number of steps. We can still construct bad trees for this modified definition, where we first

create a tree which has  $l$  leaves at height  $f^{-1}(l)$ . This construction requires only adding a polynomial height to the bad trees, hence unprovability of the theorem variant for these new trees is preserved.

There is an interesting relation between growing trees and Hydra battles in the following sense. Label every leaf  $l$  of the tree with the ordinal  $\omega^{h+1-\text{height}(l)}$ . We encode a growing tree which does not reach height  $h$  as a Hydra battle by taking the commutative sum of the labels on the leaves of the trees. This shows that a bound on the number of steps required to grow a tree of height  $h$  can be determined by studying the length of a Hydra battle starting with  $\omega^{h+1}$ .





## Chapter 4

# MacLagan's principle

Quite recently Diane MacLagan [32] has proved the following interesting theorem:

**60 Theorem.** *Every infinite sequence of monomial ideals in a polynomial ring contains an ideal that is a subset of an ideal that occurs earlier in that sequence.*

Monomial ideals play an important role in commutative algebra and algebraic combinatorics. Because it has several applications in computer algebra it is of interest to study the logical and combinatorial issues surrounding this theorem. Aschenbrenner and Pong [4] did this extensively from the viewpoint of the theory of well partial orders and they computed several related and very interesting ordinal invariants. We complement this, in particular Proposition 3.25 of that paper which concerns a finite version of MacLagan's theorem. We show that already in two variables there are bad sequences with linear complexity bounds which have non-primitive recursive lengths. We also extend this, for arbitrary  $n$ , to  $n$ -fold recursive lengths with higher numbers of variables. This is somewhat surprising because upper bounds for the lengths of increasing chains of ideals with linear complexity bounds that arise from the similarly shaped Hilbert Basis theorem are primitive recursive for any fixed number of variables (Moreno Socías [42]).

The consequence of this result is that finite MacLagan is one of the rare examples of finite theorems arising from practice that are not provable in  $\text{I}\Sigma_2$ . The proof in this chapter is a much cleaned up version compared to our proof in [37]. Additionally we determine the transition threshold for MacLagan's theorem.

## 4.1 Preliminaries

In this chapter we take arbitrary field  $K$  and examine ideals in the polynomial ring  $K[X_d, \dots, X_0, Y]$ .

**61 Definition.** *A monomial is a polynomial of the form  $X_d^{i_d} \dots X_0^{i_0} Y^j$ . A monomial ideal is an ideal that is generated by monomials.*

We denote an ideal that is generated by a set  $G$  of generators with  $\langle G \rangle$ .

**62 Definition.**

1. The degree of a monomial is the total degree:

$$\deg(X_d^{i_d} \dots X_0^{i_0} Y^j) = i_d + \dots + i_0 + j.$$

2. The degree of a set  $G$  of monomials is the maximum of the degrees of the elements of that set:  $\deg(G) = \max\{\deg(m) : m \in G\}$ .
3. The degree of a monomial ideal  $I$  is the smallest degree that is needed to be able to generate it with monomials:  $\deg(I) = \min\{\deg(G) : I = \langle G \rangle\}$ .

We miniaturise theorem 60 to obtain a Friedman-style finite version similar to the finite Kruskal's theorem:

**63 Theorem.** *For every  $l$  there exists an  $M$  such that for every sequence  $I_0, \dots, I_M$  of monomial ideals in  $K[X_d, \dots, X_0, Y]$ , with  $\deg(I_i) \leq l + i$  for all  $i \leq M$ , there exist  $i < j \leq M$  with  $I_i \supseteq I_j$ .*

Introducing a parameter  $f: \mathbb{N} \rightarrow \mathbb{N}$  (if  $f = \text{id}$  then we obtain the same theorem):

**64 Theorem** ( $\text{MM}_f$ ). *For every  $l$  there exists an  $M$  such that for every sequence  $I_0, \dots, I_M$  of monomial ideals in  $K[X_d, \dots, X_0, Y]$ , with  $\deg(I_i) \leq l + f(i)$  for all  $i \leq M$ , there exist  $i < j \leq M$  with  $I_i \supseteq I_j$ .*

Both theorems are proved using MacLagan's theorem and König's lemma. The proof of MacLagan's theorem can be found in both [32] and [4].

We use  $M_d^f(l)$  to denote the least  $M$  from the latter theorem. It may be of interest to note that  $\text{MM}$  restricted to ideals with one generator is in fact Dickson's lemma.

## 4.2 Lower bounds for the identity function

We will first examine  $\text{MM}_{\text{id}}$ . Compared with our proof in [37] we have removed the inconvenient step of constructing special descending sequences of ordinals and encoded those directly in the monomial ideals. Furthermore intermediate sequences have been removed, greatly simplifying the proof (especially the book-keeping of the degrees of the ideals in sequences).

**65 Theorem.**  $\text{IS}_2 \not\vdash \text{MM}_{\text{id}}$

We call a sequence  $I_0, \dots, I_R$  with  $\deg(I_i) \leq l + i$  for all  $i \leq R$  bad if there does not exist  $i < j \leq R$  with  $I_i \supseteq I_j$ . We call a sequence of sets of generators bad if the sequence of ideals generated by those sets is bad. Such a sequence shows that  $R < M_d^{\text{id}}(l)$ .

We will associate with each ordinal  $\alpha < \omega^{d+1}$  some monomials and number  $h_\alpha$ . We construct sequences of sets of generators consisting of monomials associated with ordinals  $\leq \alpha$  that show  $F_\alpha(l) < M_d^{\text{id}}(l + h_\alpha)$ . With the risk of confusing the reader we leave out many brackets in the definitions of sequences and sets of generators. We will also be sloppy in denoting ideals with their generators.

Given:

$$\alpha = \omega^d \cdot n_d + \dots + \omega^0 \cdot n_0,$$

we associate with  $\alpha$  the set of monomials of the form:

$$X_d^{n'_d} \dots X_0^{n'_0} Y^m,$$

where  $n'_i \leq 2 \cdot n_i + 1$ ,  $m \in \mathbb{N}$  and the number  $h_\alpha = 2 \cdot n_d + \dots + 2 \cdot n_0 + d + 1$ . Because the existence of such a sequence implies:

$$F_{\omega^d}(l) < M_d^{\text{id}}(l + 3 + d + 1),$$

for all  $d$  this suffices to prove theorem 65.

The bad sequences are defined by recursion on  $\alpha$ :

- For  $\alpha = 0$  we take the following sequence:

$$\begin{aligned} \text{Seq}(\alpha, l)_0 &= X_0, \\ \text{Seq}(\alpha, l)_{1+i} &= Y^{l+1-i} \quad \text{for } 0 \leq i \leq l + 1. \end{aligned}$$

- For  $\alpha + 1$  we start the construction with, for  $0 \leq i \leq l$ :

$$\text{Seq}(\alpha + 1, l)_i = X_d^{2 \cdot n_d + 1} \dots X_1^{2 \cdot n_1 + 1} X_0^{2 \cdot n_0 + 3} Y^{l-i}.$$

Continuing with, for  $F_\alpha^0(l) + \dots + F_\alpha^j(l) < i \leq F_\alpha^0(l) + \dots + F_\alpha^{j+1}(l)$  and  $0 \leq j \leq l$ :

$$\text{Seq}(\alpha + 1, l)_i = X_d^{2 \cdot n_d + 1} \dots X_1^{2 \cdot n_1 + 1} X_0^{2 \cdot n_0 + 2} Y^{l-j}, b_i \cdot Y^{l+1},$$

where  $b_i = \text{Seq}(\alpha, F_\alpha^j(l))_{i - F_\alpha^0(l) - \dots - F_\alpha^j(l)}$  and  $b_i m$  denotes the set consisting of the elements from  $b_i$  multiplied by monomial  $m$ .

- For limit  $\alpha = \omega^d \cdot n_d + \dots + \omega^j \cdot (n_j + 1) + \dots + \omega^0 \cdot n_0$  (where  $j > 0$  and  $n_{j-1}, \dots, n_0 = 0$ ) we take the sequence defined as follows, for  $0 \leq i \leq l$ :

$$\text{Seq}(\alpha, l)_i = X_d^{2 \cdot n_d + 1} \dots X_{j+1}^{2 \cdot n_{j+1} + 1} X_0^{2 \cdot n_j + 3} Y^{l-i}.$$

For  $0 \leq i \leq 2l$ :

$$\text{Seq}(\alpha, l)_{l+i+1} = X_d^{2 \cdot n_d + 1} \dots X_{j+1}^{2 \cdot n_{j+1} + 1} X_0^{2 \cdot n_j + 2} Y^{2l-i}.$$

For  $0 < i \leq F_{\alpha[l]}(l)$ :

$$\text{Seq}(\alpha, l)_{3l+i+1} = \text{Seq}(\alpha[l], l)_i.$$

*Claim:* The sequences  $\text{Seq}(\alpha, l)$  show  $F_\alpha(l) < M_d^{\text{id}}(l + h_\alpha)$ .

*Proof:* According to the definition the sequences  $\text{Seq}(\alpha, l)$  are long enough. First we show, using induction on  $\alpha$ , that the degrees are bounded linearly:

- $\alpha = 0$ :  $\deg(X_0) = 1$  and, if  $0 < i \leq l + 2$  then we have:

$$\deg(\text{Seq}(\alpha, l)_i) = l + 2 - i \leq l \leq l + h_0 + i.$$

- $\alpha + 1$ : if  $0 \leq i \leq l$  then:

$$\deg(\text{Seq}(\alpha, l)_i) = 2 \cdot n_d + \dots + 2 \cdot n_0 + d + 3 + l - i \leq l + h_{\alpha+1} + i.$$

If  $F_\alpha^0(l) + \dots + F_\alpha^j(l) < i \leq F_\alpha^0(l) + \dots + F_\alpha^{j+1}(l)$  and  $0 \leq j \leq l$  then:

$$\deg(\text{Seq}(\alpha, l)_i) \leq \max\{h_{\alpha+1}, \deg(b_i) + l + 1\} \leq l + h_{\alpha+1} + i,$$

where the second inequality is obtained from the definition of  $b_i$  and the induction hypothesis.

- Limit  $\alpha$ : If  $0 \leq i \leq l$  then:

$$\deg(\text{Seq}(\alpha, l)_i) \leq l + h_\alpha - i \leq l + h_\alpha + i.$$

If  $l < i \leq 3l + 1$  then:

$$\deg(\text{Seq}(\alpha, l)_i) \leq l + h_\alpha + (l + 1 - i) \leq l + h_\alpha + i.$$

If  $3l + 1 < i \leq 3l + 1 + F_{\alpha[l]}(l)$  then:

$$\deg(\text{Seq}(\alpha, l)_i) = \deg(\text{Seq}(\alpha[l], l)_{i-3l-1}) \leq l + h_{\alpha[l]} + i - 3l - 1,$$

where the latter inequality results from the induction hypothesis. Notice that  $h_{\alpha[l]} = h_\alpha + 2l$ , hence:

$$\deg(\text{Seq}(\alpha, l)_i) \leq l + h_\alpha + 2l + i - 3l - 1 \leq h_\alpha + i - 1 \leq l + h_\alpha + i.$$

We still need to prove that these sequences are bad. We will use the fact that a monomial is an element of a monomial ideal if and only if one of the generators divides that monomial and that in the construction the generators in each sequence consist of monomials which are associated with ordinals  $\leq \alpha$  exclusively.

- $\alpha = 0$ : Notice that  $X_0$  does not divide  $Y^a$  and if  $i < j \leq l + 1$  then  $Y^{l+1-i}$  does not divide  $Y^{l+1-j}$ .
- $\alpha + 1$ : The generators  $X_d^{2 \cdot n_d + 1} \dots X_1^{2 \cdot n_1 + 1} X_0^{2 \cdot n_0 + 1 + a} Y^b$  ( $a = 1, 2$ ) do not divide any monomials that are associated with  $\beta \leq \alpha$ . Indeed, if such a generator divided such a monomial we would have  $2 \cdot n_0 + 1 + a \leq 2 \cdot n_0 + 1$ . Hence if

$$F_\alpha^0(l) + \dots + F_\alpha^j(l) < i_0 < i_1 \leq F_\alpha^0(l) + \dots + F_\alpha^{j+1}(l),$$

then any generator of  $\text{Seq}(\alpha + 1, l)_{i_1}$  that is associated with ordinals  $\leq \alpha$  is not divided by  $X_d^{2 \cdot n_d + 1} \dots X_1^{2 \cdot n_1 + 1} X_0^{2 \cdot n_0 + 2} Y^{l-j}$ . By induction hypothesis those generators (from  $b_{i_1} Y^{l+1}$ ) can also not be divided by the other elements of  $\text{Seq}(\alpha + 1, l)_{i_0}$ .

If

$$i_0 \leq F_\alpha^0(l) + \dots + F_\alpha^j(l) < i_1,$$

then the generator  $X_d^{2 \cdot n_d + 1} \dots X_1^{2 \cdot n_1 + 1} X_0^{2 \cdot n_0 + 1 + a} Y^b$  is not divided by any element of  $\text{Seq}(\alpha + 1, l)_{i_0}$ .

- Limit  $\alpha$ : Again, if  $i_0 < 3l + 1 < i_1$  then the generators in  $\text{Seq}(\alpha, l)_{i_1}$  are associated with ordinals  $< \alpha$ , hence cannot be divided by any generator from  $\text{Seq}(\alpha, l)_{i_0}$ . If  $3l + 1 < i_0 < i_1$  then the induction hypothesis delivers the same fact as does the definition of  $\text{Seq}(\alpha, l)$  when  $i_0 < i_1 \leq l + 1$  or  $l + 1 < i_0 < i_1 \leq 3l + 1$ .

This ends the proof of the claim thus finishing the proof of theorem 65.

□

As a side note we have also shown that:

$$F_{\omega^1}(l) < M_d^{\text{id}}(l + 5),$$

hence:

**66 Corollary.**  $\text{IS}_1 \not\vdash \text{MM}^1$ , where  $\text{MM}^1$  is MM limited to two variables.

### 4.3 Lower bounds for other parameter values

In this section we modify bad sequences for the identity into bad sequences for lower parameter values  $f$ , showing that  $M^f$  again is unbounded in the multiply recursive functions. The first step of the modification is to, given a sequence:

$$I_0, \dots, I_M,$$

define the new sequence:

$$I_{f(0)}, \dots, I_{f(M)}.$$

This new sequence will have identical elements, but we correct this by modifying the ideals using a constant number of extra variables. To be able to do this we will need to estimate the number  $c'$  sufficiently large such that

$$\#\{i : f(i) = f(j)\} \leq M_{c'}^0(j).$$

For this reason we start with studying  $M^0$ .

**67 Lemma.**  $M_0^0(2j + 2) \geq 2^j$ .

*Proof:* We construct sequences that show this using recursion on  $j$ .

For  $j = 0$  we take sequence  $m_0 = Y^2, m_1 = Y$ .

Given sequence  $a_0, \dots, a_{2^j}$  for  $j$ , take for  $j + 1$  the sequence defined by:

$$m_i = \begin{cases} X_0^{j+1}Y^1, a_iY^2 & \text{if } 0 \leq i \leq 2^j, \\ X_0^{j+1}Y^0, a_{i-2^j-1}Y^2 & \text{if } 2^j + 1 \leq i \leq 2^{j+1}. \end{cases}$$

□

**68 Lemma.**  $M_{d+2}^0(j + c) \geq M_d^0(j)^{c+1}$ .

*Proof:* We construct sequences which show this. Taking a bad sequence  $a_0, \dots, a_M$ , the elements of the new sequence will look like:

$$X_{d+2}^c X_{d+1}^0 a_{i_0}, \dots, X_{d+2}^{c-j} X_{d+1}^j a_{i_j}, \dots, X_{d+2}^0 X_{d+1}^c a_{i_c}.$$

The main idea is that the generators of the ideals in the new sequence get separated into 'tracks', where due to the part  $X_{d+2}^{c-j} X_{d+1}^j$  the generators from different tracks cannot divide each other. Hence if we change in this set of generators  $a_{i_j}$  into  $a_{i_j+1}$  the ideal generated by this new set is not a subset of the original ideal. Using this we construct the new sequences by recursion on  $c$ .

For  $c = 0$  we take  $m_i = a_i$ .

Given sequence  $b_0, \dots, b_N$  for  $c$  and  $0 \leq i \leq M \cdot N$  we take:

$$m_i = X_{d+2}^c a_{\frac{i}{N}}, X_{d+1} b_{i \% N}.$$

□

**69 Lemma.**  $M_{2c}^0(2^{c+1}(j+1)) \geq 2^{j^{c+1}}$ .

*Proof:* Combine the previous two lemmas, starting with the first for  $c = 0$ , for the induction step we use the latter.

□

With the constructions so far it is not possible to obtain double-exponential lengths of such sequences, an attempt to do so would require using a non-constant number of variables. We will later see that this is not possible using any construction due to the upper bounds on  $M^0$ . We take this ‘highest possible’ estimate of  $M^0$  to prove unprovability for the following ‘low’ parameter.

**70 Theorem.** If  $f_c(i) = \sqrt[c]{\log(i)}$  then:  $\text{IS}_2 \not\vdash \text{MM}_{f_c}$ .

*Proof:* We use lemma 69 to convert bad sequences for identity into bad sequences for  $f_c$ . Together with theorem 65 the following is sufficient to prove this theorem:

$$M_{d+2c+3}^{f_c}(l) \geq M_d^{\text{id}}(l),$$

for  $l \geq 2^{(c+4)^2}$ . Our building blocks are bad sequences  $a_0, \dots, a_M$  ( $d+2$  variables) and  $b(i)_0, \dots, b(i)_{2(i+1)^c}$  ( $2c+2$  new variables and  $i > 2^{(c+1)^2}$ ). The new sequence is:

$$m_i = \begin{cases} X_{d+2c+3}^{2^{(c+1)^2}-i} & \text{if } i \leq 2^{(c+1)^2} \\ a_{f_c(i)}, b(f_c(i))_{i-2^{f_c(i)}c} & \text{otherwise.} \end{cases}$$

The fact that this is a bad sequence and the bounds on the degrees are inherited from the original sequences.

□

**71 Theorem.** If  $f(i) = \sqrt[F_{\omega\omega}^{-1}(i)]{\log(i)}$  then:  $\text{IS}_2 \not\vdash \text{MM}_f$ .

*Proof:* We show:

$$M_{3l+3}^f(2^{(l+4)^2} + 2l + 4) \geq F(l).$$

Assume for a contradiction that  $M = M_{3l+3}^f(2^{(l+4)^2} + 2l + 4) < F(l)$ . For  $i \leq M$  we know that  $F^{-1}(i) \leq l$ , in other formulas that  $\sqrt[F^{-1}(i)]{\log(i)} \geq \sqrt[l]{\log(i)} = f_l(i)$ ,

using notation from the previous theorem. The estimates from theorems 65 and 70 deliver:

$$\begin{aligned} M &\geq M_{3l+3}^{f_l}(2^{(l+4)^2} + 2l + 4) \\ &\geq M_l^{\text{id}}(2l + 4) \\ &\geq F_{\omega^l}(l) = F(l) \end{aligned}$$

which contradicts our assumption. □

## 4.4 Upper bounds

For the upper bounds we use a simple counting of monomials, notice first that:

**72 Lemma.**  $\text{EFA} \vdash \text{MM}_c$  for constant function  $c$ .

*Proof:* The number of ideals of degree less than  $l + c$  is bounded by the number of sets of monomials of degree less than  $l + c$ . This number has (rough) upper bound  $2^{(l+c+1)^{d+2}}$ . Hence, by pigeonhole principle,  $M_d^c(l) \leq 2^{(l+c+1)^{d+2}} + 1$ . □

Using essentially the same argument we obtain the following result:

**73 Lemma.** If  $B$  is an increasing multiply recursive function and

$$f(i) = {}^{B^{-1}(i)}\sqrt{\log(i)}$$

then:

$$\text{IS}_2 \vdash \text{MM}_f.$$

*Proof:* Assume without loss of generality that  $B(l) > 2^{l+2}$ . We show for  $l > d + 2$ :

$$M_d^f(l) < 2^{B(l)^{d+2}}.$$

Take  $R = 2^{B(l)^{d+2}}$  and any sequence of monomial ideals  $I_0, \dots, I_R$  with  $\deg(I_i) \leq l + f(i)$ . In this case we have:

$$\deg(I_i) \leq l + {}^{B^{-1}(R)}\sqrt{\log(R)} \leq l + B(l)^{\frac{d+2}{l}} \leq l + B(l).$$

So by the upper bounds from the proof of lemma 72:

$$M_d^f(l) \leq M_d^{B(l)}(l) \leq 2^{(l+B(l)+1)^{d+1}} + 1 < 2^{2^{d+1}B(l)^{d+1}} + 1 < 2^{B(l)^{d+2}}.$$

□



## Chapter 5

# Unordered Kanamori–McAloon

The Kanamori–McAloon theorem is the second of the Ramsey-like unprovability results in PA. It may look more natural than the Paris–Harrington theorem because it removes the largeness condition from the size of the set for which the colouring is constant. The unprovability proof for KM makes essential use of the standard  $<$ -relation and one might wonder if it is possible to find a strong principle which does not depend so intrinsically on the less than relation.

An interesting approach to this question can be obtained from a recent paper by Richer [39] about unordered canonical Ramsey numbers and their asymptotic classification. It is quite natural to extend Richer’s approach to the context of strong Ramsey principles and we will do so here for dimension 2 and for unlimited dimensions.

**74 Definition.** We call a colouring  $C: [R]^d \rightarrow \mathbb{N}$  *f-regressive* if  $C(x) \leq f(\min x)$  for all  $x \in [R]^d$ .

**75 Definition.**

1. A set  $H \subseteq R$  is *homogeneous* if  $C(x) = C(y)$  for all  $x, y \in [H]^d$ .
2. A set  $H \subseteq R$  is *min-homogeneous* if  $C(x) = C(y)$  for all  $x, y \in [H]^d$  with  $\min x = \min y$ .
3. A set  $H \subseteq R$  is *min- $\prec$ -homogeneous* if  $C(x) = C(y)$  for all  $x, y \in [H]^d$  with  $\min_{\prec} x = \min_{\prec} y$ .

**76 Theorem** ( $\text{uKM}_f$ ). For every  $d, m$  there exists an  $R$  such that for every *f-regressive* colouring  $C: [R]^d \rightarrow \mathbb{N}$  there exists an  $H$  of size  $m$  with linear order  $\prec \subseteq H^2$  which is *min- $\prec$ -homogeneous* for  $C$ .

*Proof:* This is a direct consequence of  $\text{KM}_f$ .

□

**77 Definition.** We denote the  $R$  obtained from this theorem  $\text{uKM}_f^d(m)$ . If  $d = 2$  we use  $\text{uKM}_f(m)$ . If the range of the colourings is  $[l, R]^d$  we use  $\text{uKM}_f^d(l, m)$ .

If we have fixed  $d$ , we denote  $\text{uKM}_f$  for this  $d$  with  $\text{uKM}_f^d$ . We immediately have

the following results from [8].

**78 Theorem.**  $\text{IS}_1 \vdash \text{uKM}_f^2$  if  $f$  is:

1. a constant function,
2.  $i \mapsto \log i$  and
3.  $i \mapsto {}^{A_d^{-1}}\sqrt[i]{i}$ .

## 5.1 Unprovability in dimension 2

For the lower bounds there does not exist such an easy proof. We modify the proofs for  $\text{KM}_f$  from [8] and [26] to suit the problem that allowing differing orderings on  $H$  gives.

**79 Definition.**

$$\begin{aligned} A_0^f(i) &= i + 1 \\ A_{n+1}^f(i) &= (A_n^f)^{f(i)}(i) \\ A^f(i) &= A_i^f(i) \end{aligned}$$

With an abuse of notation we denote  $A_n^{\sqrt[s]{\phantom{x}}}$  and  $A^{\sqrt[s]{\phantom{x}}}$  with  $A_n^s$  and  $A^s$ .

**80 Definition.**  $R_c^d(i)$  is the least  $R$  such that for each colouring  $C: [R]^d \rightarrow c$  there exists  $Y$  of size  $i$  that is homogeneous for  $C$ .

The following is a fact from Ramsey theory [20]:

**81 Lemma.**  $R_c^d(i)$  is primitive recursive.

We give a lower bound for  $\text{uKM}_f$  which ensures that it is Ackermannian. This proof rests on two ideas. The first idea is the use of the particular colourings similar to proofs of the ordered  $\text{KM}_f$ . The second idea is increasing the ‘space’ in the  $(\min_{\prec})$ -homogeneous sets for those colourings in order to solve the problem caused by allowing any linear order to determine  $\min_{\prec}$ -homogeneity. Fix  $s \in \mathbb{N}$ .

**82 Lemma** (lower bound for roots). *Let  $f: i \mapsto \sqrt[s]{i}$ , then:*

$$\text{uKM}_f(R_c^2(m+4)) \geq A_{c+1}^s(m)$$

for all  $c, m \in \mathbb{N}$ .

*Proof:* Take  $k = R_c^2(m + 4)$  and  $R = \text{uKM}_f(k)$ . Define a colouring  $C$  on  $R$  as follows for  $x < y$ :

$$C(x, y) = \begin{cases} 0 & \text{if } A_{c+1}^s(x) \leq y, \\ l & \text{else,} \end{cases}$$

where  $l$  is such that for the smallest  $p$  for which  $A_{p+1}^s(x) > y$  we have  $A_p^{s(l)}(x) \leq y < A_p^{s(l+1)}(x)$ .

Taking  $p$  for  $x, y$  as above, define colouring  $D$  of  $R$  for  $x < y$ :

$$D(x, y) = \begin{cases} 0 & \text{if } A_{c+1}^s(x) \leq y, \\ p & \text{else.} \end{cases}$$

Note that  $C$  is  $f$ -regressive (because  $A_p^{s(f(x))}(x) = A_{p+1}^s(x)$ ). Let  $H \subseteq R$  of size  $k$  with order  $\prec$  be  $\min_{\prec}$ -homogeneous for  $C$ , then by definition of  $k$  there exists  $Y \subseteq H$  of size  $m + 4$  which is  $D$ -homogeneous. Enumerate such a  $Y$  with a strictly  $<$ -increasing sequence  $Y = \{y_1, \dots, y_m, x, y, z, z'\}$ . Then we have the following cases for the relative  $\prec$ -ordering of  $x, y, z, z'$ :

1.  $x \prec y, x \prec z$

*Claim:*  $A_{c+1}^s(x) \leq y$ .

Assume for a contradiction that  $A_{c+1}^s(x) > y$ , by definition of  $C$  we get  $C(x, y) = l \neq 0$ . Hence (by  $\min_{\prec}$ -homogeneity of  $H$ )  $C(x, y) = C(x, z) = l$ . By definition of  $D$  and  $D$ -homogeneity of  $Y$  we also get  $D(y, z) = D(x, y) = p \neq 0$ .

So the definition of  $C$  gives us:

$$A_p^{s(l)}(x) \leq y < A_p^{s(l+1)}(x)$$

and

$$A_p^{s(l)}(x) \leq z < A_p^{s(l+1)}(x),$$

that of  $D$  delivers:

$$A_p^s(y) \leq z.$$

Combining these inequalities, taking note that  $A_p^s$  is increasing, we get the contradiction:

$$z < A_p^{s(l+1)}(x) = A_p^s(A_p^{s(l)}(x)) \leq A_p^s(y) \leq z.$$

2.  $z \prec x, z \prec y$

*Claim:*  $A_{c+1}^s(x) \leq z$ .

Assume  $A_{c+1}^s(x) > z$ , by definition and  $\min_{\prec}$ -homogeneity of  $C$  we have  $C(x, z) = C(y, z) = l \neq 0$ , by definition and homogeneity of  $D$  we get:  $D(x, y) = D(x, z) = p$ . This gives us inequalities:

$$A_p^{s(l)}(x) \leq z < A_p^{s(l+1)}(x),$$

$$A_p^{s(l)}(y) \leq z < A_p^{s(l+1)}(y)$$

and

$$A_p^s(x) \leq y.$$

Combining these we get:

$$z < A_p^{s(l+1)}(x) = A_p^{s(l)}(A_p(x)) \leq A_p^{s(l)}(y) \leq z.$$

3.  $y \prec x, y \prec z$ , we distinguish two possibilities:

(a)  $y \prec z'$

*Claim:*  $A_{c+1}^s(y) \leq z$ .

Assume  $A_{c+1}^s(y) > z$ , then  $C(y, z) = C(y, z') = l \neq 0$  and  $D(z, z') = D(y, z') = p$ . So we have inequalities:

$$A_p^{s(l)}(y) \leq z < A_p^{s(l+1)}(y),$$

$$A_p^{s(l)}(y) \leq z' < A_p^{s(l+1)}(y)$$

and

$$A_p^s(z) \leq z'.$$

Combining these:

$$z' < A_p^s(A_p^{s(l)}(y)) \leq A_p^s(z) \leq z'.$$

(b)  $z' \prec y$

*Claim:*  $A_{c+1}^s(x) \leq z'$ .

Assume  $A_{c+1}^s(x) > z'$ , then  $C(x, z') = C(y, z') = l \neq 0$  and  $D(x, y) = D(x, z') = p$ . So we have:

$$A_p^{s(l)}(x) \leq z' < A_p^{s(l+1)}(x),$$

$$A_p^{s(l)}(y) \leq z' < A_p^{s(l+1)}(y)$$

and

$$A_p^s(x) \leq y.$$

Combining these:

$$z' < A_p^{s(l)}(A_p^s(x)) \leq A_p^{s(l)}(y) \leq z'.$$

Examining the cases above allows us to conclude  $A_{c+1}^s(x) \leq z'$ . But:

$$A_{c+1}^s(m) \leq A_{c+1}^s(y_m) \leq A_{c+1}^s(x) \leq z' \in Y \subseteq H \subseteq R.$$

So we finally have:

$$A_{c+1}^s(m) \leq R.$$

□

**83 Corollary.**  $\text{IS}_1 \not\vdash \text{uKM}_{\mathcal{E}'}^2$ .

*Proof:* Observe  $A_n(i) \leq A_{n+2s^2+1}^s(i)$  for any  $i \geq 4^s$  (See [8], corollary 4.3). Hence no primitive recursive function can bound  $\text{uKM}_{\mathcal{E}'}^2$ .

□

This result can be sharpened as follows:

**84 Theorem.**  $\text{IS}_1 \not\vdash \text{uKM}_f^2$  where  $f(i) = {}^{A^{-1}(i)}\sqrt{i}$ .

*Proof:* We claim:

$$R = \text{uKM}_f(R_{m+2m^2}^2(4^m + 5)) \geq A(m).$$

Assume, for a contradiction, that  $R < A(m)$ . If  $i \leq R$  then  $A^{-1}(i) \leq m$ , so  $f(i) \geq {}^m\sqrt{i}$ . Hence:

$$\begin{aligned} R &\geq \text{uKM}_{\mathcal{E}'}(R_{m+2m^2}^2(4^m + 4)) \\ &\geq A_{m+m^2+1}^m(4^m) \\ &> A_m(m) = A(m), \end{aligned}$$

which is in contradiction with our assumption. The second inequality is due to lemma 82.

□

## 5.2 Unprovability for unlimited dimension

We use the classic method to show that there exists a model of  $\text{PA} + \neg \text{uKM}_{\log^n}$ . Additionally we give a sharpening of the transition result using a combination of a recursion theoretic method and model construction. The first proof we show has at its core the Kanamori–McAloon proof with the compactness arguments and the phase transition combinatorics handled in one go by using nonstandard numbers. The main tool we use will be a countably infinite set of indiscernibles.

**85 Definition.** We call  $c_0 < c_1 < c_2 < \dots$  *indiscernibles of a model  $M$*  if for every  $\varphi \in \Delta_0$ ,  $i_0 < i_1 < \dots < i_n$ ,  $j_0 < j_1 < \dots < j_n$ :

$$M \models \forall p < c_{i_0} (\varphi(p, c_{i_1}, \dots, c_{i_n}) \leftrightarrow \varphi(p, c_{j_1}, \dots, c_{j_n})).$$

To obtain indiscernibles we will be using some recursive functions that involve the satisfaction of all standard  $\Delta_0$  formulas in a nonstandard model  $M$  of  $\text{IS}_1$ . This is possible because these formulas can be enumerated primitive recursively, hence any enumeration of the first  $e$  such formulas using such enumeration function, where  $e$  is nonstandard, will include the standard formulas. The functions themselves will be well-defined internally in the model because there exists relation  $\text{Sat}_{\Delta_0}$  such that for all  $\Delta_0$ -formulas  $\varphi$ :

$$\text{IS}_1 \vdash \forall x (\varphi(x) \leftrightarrow \text{Sat}_{\Delta_0}(\ulcorner \varphi \urcorner, x)).$$

More details on this can be found in [24].

**86 Theorem.** For every  $n \in \mathbb{N}$ ,  $\text{PA} \not\vdash \text{uKM}_{\log^n}$ .

*Proof:* We closely follow Bovykin's proof for unprovability of KM [5] and will indicate where the proof has changed significantly. Fix an encoding such that every subset of  $[0, a]$  has code below  $2^{a+1} + 1$ , denote the set coded by  $a$  with  $X(a)$ . Start with nonstandard model  $M \models \text{IS}_1$  and elements  $a > e > \mathbb{N}$ . Let  $\varphi_1(z, x_1, \dots, x_e), \dots, \varphi_e(z, x_1, \dots, x_e)$  be the first  $e$   $\Delta_0$ -formulas in at most the free variables shown, where the formulas of standard size are included in this list. Take  $r(e) = R_{e+2}^{2e+1}(2^{n+3} \cdot e)$  and the least  $b$  such that for every  $\log^n$ -regressive  $g: [a, b]^{2^{n+1} \cdot e+2} \rightarrow M$  there exists  $H \subseteq [a, b]$  of size  $r(e)$  with linear order  $\prec$  which is  $\min_{\prec}$ -homogeneous for  $g$ .

Define  $f: [a, b]^{2e+1} \rightarrow e+2$  to be

$$f(c, \bar{d}_1, \bar{d}_2) = \min\{i : \exists p < c. \varphi_i(p, \bar{d}_1) \not\leftrightarrow \varphi_i(p, \bar{d}_2)\}$$

and equal to  $e+1$  if this is undefined. Define the following regressive function:

$$h(c, \bar{d}_1, \bar{d}_2) = \min\{p < c : \varphi_j(p, \bar{d}_1) \not\leftrightarrow \varphi_j(p, \bar{d}_2)\},$$

when  $j = f(c, \bar{d}_1, \bar{d}_2) < e+1$  and equal to  $c$  otherwise.

Here starts the definition of a  $\log^n$ -regressive function  $l$ . Let  $c < \bar{d}_1 < \dots < \bar{d}_{2^{n+1}} < s$  be given. Take for  $1 \leq k \leq 2^n$ :

$$p_k^{n+1} = h(c, \bar{d}_{2k-1}, \bar{d}_{2k})$$

and for every  $1 \leq i \leq n$ ,  $1 \leq k \leq 2^{i-1}$ :

$$p_k^{i-1} = \min X(p_{2k-1}^i) \ominus X(p_{2k}^i),$$

where  $\ominus$  denotes the symmetric difference, where it equals  $\log^{n-i+1} c$  when this difference is empty. Using this take:

$$l(c, \bar{d}_1, \dots, \bar{d}_{2^{n+1}}, s) = p_1^1.$$

Note that thanks to our choice of coding this function is  $\log^n$ -regressive. Furthermore an extra dimension is used compared to the original proof. For clarity we used  $p_j^i$  to denote  $p_j^i(c, \bar{d}_1, \dots, \bar{d}_{2^{n+1}}, s)$  in this definition.

**From this point the proof is changed:**

Take  $H \subseteq [a, b]$  of size  $2^{n+3} \cdot e$  and order  $\prec$  such that  $H$  is  $\min_\prec$ -homogeneous for  $l$  and homogeneous for  $f$ . Take  $h = \min_\prec H$  and:

$$\hat{H} = \{x \in H : x \geq h\}$$

if  $h$  lies in the first half of  $H$  and:

$$\hat{H} = \{x \in H : x \leq h\}$$

otherwise. Note that  $l(\min \hat{H}, \dots, \max \hat{H})$  is constant by  $l$ - $\min_\prec$ -homogeneity of  $H$  and denote this value with  $p$ . Take elements  $c = \min \hat{H} < \bar{d}_1 < \dots < \bar{d}_{3 \cdot 2^n} < \max \hat{H} = s$  from  $\hat{H}$ .

*Claim 1:* For all  $1 < i \leq n+1$  we have that  $\bar{x} \mapsto p_1^{i-1}(c, \bar{x}, \bar{d}_{2^{n-i+4}+1}, \dots, \bar{d}_{3 \cdot 2^n}, s)$  has value  $\log^{n-i+1}(c)$  on  $\hat{H}_{<\bar{d}_{2^{n-i+4}+1}}$ , hence  $\bar{y} \mapsto p_1^i(c, \bar{y}, \bar{d}_{2^{n-i+3}+1}, \dots, \bar{d}_{3 \cdot 2^n}, s)$  is constant on  $\hat{H}_{<\bar{d}_{2^{n-i+3}+1}}$ . We show this with induction on  $i$ :

For  $i = 2$  we know  $p_1^1$  is constant. Take:

$$p_1 = p_1^2(c, \bar{d}_1, \dots, \bar{d}_{2^n}, \bar{d}_{2^{n+1}}, \dots, \bar{d}_{2^{n+1}}, s),$$

$$p_2 = p_2^2(c, \bar{d}_1, \dots, \bar{d}_{2^n}, \bar{d}_{2^{n+1}}, \dots, \bar{d}_{2^{n+1}}, s) \text{ and}$$

$$p_3 = p_2^2(c, \bar{d}_{2^{n+1}}, \dots, \bar{d}_{2^{n+1}}, \bar{d}_{2^{n+1}+1}, \dots, \bar{d}_{3 \cdot 2^n}, s).$$

Suppose that  $p = p_1^1 < \log^n(c)$ , by definition and  $p_1^1$  being constant:

$$p \in X(p_1) \Leftrightarrow p \notin X(p_2) \Leftrightarrow p \in X(p_3) \Leftrightarrow p \notin X(p_1),$$

which is a contradiction, hence  $p = \log^n(c)$ .

For the induction step assume for contradiction  $p_1^{i-1}(c, \bar{x}, \bar{d}_{2^{n-i+4}+1}, \dots, \bar{d}_{3 \cdot 2^n}, s)$  has value  $q < \log^{n-i+1}(c)$  (it is constant by induction hypothesis). Denote  $\bar{d} = \bar{d}_{2^{n-i+4}+1}, \dots, \bar{d}_{3 \cdot 2^n}, s$ . Take:

$$\begin{aligned} p_1 &= p_1^i(c, \bar{d}_1, \dots, \bar{d}_{2^{n-i+2}}, \bar{d}_{2^{n-i+2}+1}, \dots, \bar{d}_{2^{n-i+3}}, \bar{d}), \\ p_2 &= p_2^i(c, \bar{d}_1, \dots, \bar{d}_{2^{n-i+2}}, \bar{d}_{2^{n-i+2}+1}, \dots, \bar{d}_{2^{n-i+3}}, \bar{d}) \text{ and} \\ p_3 &= p_2^i(c, \bar{d}_1, \dots, \bar{d}_{2^{n-i+2}}, \bar{d}_{2^{n-i+3}+1}, \dots, \bar{d}_{2^{n-i+4}}, \bar{d}). \end{aligned}$$

Hence:

$$q \in X(p_1) \Leftrightarrow q \notin X(p_2) \Leftrightarrow q \in X(p_3) \Leftrightarrow q \notin X(p_1),$$

which is a contradiction, hence  $q = \log^{n-i+1}(c)$ .

This finishes the proof of claim 1.

*Claim 2:*  $f$  has value  $e + 1$  on  $\hat{H}$ . Assume for a contradiction that it has value  $j < e + 1$ . From claim 1 we have  $p_1^n(\bar{x}, \bar{d}) = \log c$ , hence  $q = h(c, \bar{d}_1, \bar{d}_2) = h(c, \bar{d}_1, \bar{d}_3) = h(c, \bar{d}_2, \bar{d}_3) < c$ , where the inequality is due to the value of  $f$ . Hence:

$$\varphi_j(q, \bar{d}_1) \Leftrightarrow \neg \varphi_j(q, \bar{d}_2) \Leftrightarrow \varphi_j(q, \bar{d}_3) \Leftrightarrow \neg \varphi_j(q, \bar{d}_1),$$

contradiction, finishing the proof of claim 2. The set  $\hat{H}$  provides us with a set of indiscernibles allowing construction of a model of  $\text{PA} + \neg \text{uKM}$  using an initial segment as in the Kanamori–McAloon [23] or the Bovykin [5] article.

Take  $I = \bigcup_{i \in \mathbb{N}} \{j < c_i\}$ , then  $I$  is closed under addition and multiplication: by indiscernibility it is sufficient to show that  $c_0 + c_1 < c_2$  and  $c_0 \cdot c_1 < c_2$ . Suppose for a contradiction  $c_0 + c_1 \geq c_2$ , then there exists  $p < c_0$  such that  $p + 1 + c_1 = c_2$ . But then, by indiscernibility,  $p + 1 + c_1 = c_3$  in contradiction with  $c_2 < c_3$ .

Suppose for contradiction that  $c_0 \cdot c_1 \geq c_2$ , so there exists  $p < c_0$  such that  $p \cdot c_1 < c_2 \leq (p + 1) \cdot c_1$ . But, by indiscernibility,  $p \cdot c_1 + c_1 \geq c_4$ , which is a contradiction because  $c_1$  and  $p \cdot c_1$  are less than  $c_2$  and  $c_2 + c_3 < c_4$ .

We will prove  $\Sigma_n$  induction in  $I$ , for arbitrary  $n$ , proving  $I$  is a model of PA. Let  $\psi$  be a  $\Sigma_n$ -formula and suppose that  $I \models \psi(p, \bar{y})$ . Using appropriate encoding this is equivalent to:

$$I \models \exists x_1 \forall x_2 \dots Q x_n \varphi(m, \bar{x}),$$

where  $Q$  is  $\forall$  if  $n$  is even,  $\exists$  if  $n$  is odd and  $\varphi$  is a  $\Delta_0$  formula. Because the indiscernibles are unbounded in  $I$  this is equivalent to ( $i$ 's in  $\mathbb{N}$ ):

$$\exists i_1 > i \forall i_2 > i_1 \dots Q i_n > i_{n-1}$$



such that

$$I \models \exists x_1 < c_{i_1} \forall x_2 < c_{i_2} \dots Qx_n < c_{i_n} \varphi(m, \bar{x}),$$

where  $m < c_i$ . The last line of this is a  $\Delta_0$ -formula, hence, by indiscernibility, this is equivalent to:

$$I \models \exists x_1 < c_{i+1} \forall x_2 < c_{i+2} \dots Qx_n < c_{i+n} \varphi(m, \bar{x}),$$

for some  $i$ , where  $m < c_i$ . Removing the coding:

$$I \models \exists x_1 < c_{i+1} \forall x_2 < c_{i+2} \dots Qx_n < c_{i+n} \varphi'(p, \bar{y}, \bar{x}).$$

By  $\Delta_0$  induction inherited from  $M$  there exists minimal such  $p$ , finishing the proof of the claim.

We know  $I < b$  and  $b$  was minimal such that it was a witness for uKM, hence uKM is not true in  $I$ , ending the proof of the theorem.

□

The following theorem shows a way to use this construction for sharpening the threshold in answer to a conjecture from [5]. Interestingly the proof of the sharpening differs very little from such sharpenings using recursion theory.

**87 Theorem.**  $\text{PA} \not\models \text{uKM}_f$  for  $f(i) = \log^{H_{\varepsilon_0}^{-1}(i)}(i)$ .

*Proof:* The idea of this proof is to construct the  $I$  for nonstandard  $n$  as previously. To be able to do this we simply limit ourselves to the case that the theorem is not obviously unprovable due to lower bound estimates. Take  $r(m) = R_{m+2}^{2^{m+1}}(2^{m+3} \cdot m)$ . We examine  $\text{uKM}_f^{2^{m+1} \cdot m+2}(l, r(m))$ . If  $\text{uKM}_f^{2^{m+1} \cdot m+2}(l, r(m)) \geq H_{\varepsilon_0}(l+m)$  for infinitely many  $l, m$  we are finished. We assume that for all but finitely many  $l, m$  we have:

$$\text{uKM}_f^{2^{m+1} \cdot m+2}(l, r(m)) < H_{\varepsilon_0}(l+m).$$

For such  $l, m$  we know that  $H_{\varepsilon_0}^{-1}(i) \leq l+m$  for all  $i \leq \text{uKM}_f^{2^{m+1} \cdot m+2}(l, r(m))$ . Take nonstandard model  $M$  and use overflow to obtain nonstandard  $a > e$  with  $\log^{H_{\varepsilon_0}^{-1}(i)}(i) \geq \log^{a+e}(i)$  for all  $i \leq \text{uKM}_f^{2^{e+1} \cdot e+2}(a, r(e))$ . Continue the construction from theorem 86 with these  $a, e$ , but with the nonstandard  $a+e$  instead of  $n$ . This construction will work because the function  $(n, \bar{z}) \mapsto l_n(\bar{z})$  (where  $l_n$  is the function  $l$  from the proof of theorem 86 with  $n$  fixed) is primitive recursive, hence properly defined for nonstandard  $n$ . Furthermore, the proofs of the two claims are in  $\text{IS}_1$ , hence they are also true for the nonstandard  $a+e$  instead of  $n$ .

□



## Chapter 6

# Adjacent Ramsey

In [16] Harvey Friedman introduced finite adjacent Ramsey, a series of natural Ramsey like principles not provable in PA. We show that, like the Paris–Harrington and Kanamori–McAloon theorems limiting the dimension of one of those results in unprovability in certain fragments of PA, with the surprising difference that dimension  $k$  already results in unprovability in  $\text{I}\Sigma_k$  where for the other two theorems one needs dimension  $k + 1$ . We also classify phase transitions for this adjacent Ramsey. Furthermore we examine whether the proof of unprovability can be adapted to PH and KM.

## 6.1 Introduction

Our starting point is one of Friedman’s adjacent Ramsey theorems (theorem D in [16]), which is a very natural unprovable statement at the level of PA in the sense that it does not use the language of Ramsey theory ( $d$ -element subsets, homogeneous, min-homogeneous) whilst remaining easily stated.

**88 Theorem** (adjacent Ramsey). *For every function  $C: \mathbb{N}^k \rightarrow \mathbb{N}^r$  there exist  $x_1 < \dots < x_{k+1}$  such that  $C(x_1, \dots, x_k) \leq C(x_2, \dots, x_{k+1})$ .*

This theorem can be proved using a combination of infinite Dickson’s lemma and infinite Ramsey. Furthermore Friedman showed that, in  $\text{RCA}_0$ , adjacent Ramsey is equivalent to the well ordering of  $\varepsilon_0$ . We show that a finite version of this theorem is not provable in PA and classify the phase transition for this theorem. Inspired by results on other Ramsey-like principles, we show that the version with restricted dimensions is not provable in certain fragments of PA and classify transitions for those cases as well. To obtain a finite version of the adjacent Ramsey theorem we use the following definition (this notion plays a similar role as regressiveness does in the Kanamori–McAloon theorem):

**89 Definition.** *A function  $C: R^k \rightarrow \mathbb{N}^r$  is limited if  $\max C(x) \leq \max x$ .*

**90 Theorem** (finite adjacent Ramsey, AR). *For every  $k, r$  there exists  $R$  such that for every limited function  $C: R^k \rightarrow \mathbb{N}^r$  there are  $x_1 < \dots < x_{k+1} < R$  with  $C(x_1, \dots, x_k) \leq C(x_2, \dots, x_{k+1})$ .*

**91 Theorem** (Friedman).  $\text{PA} \not\vdash \text{AR}$ .

**92 Definition.** A function  $C: R^k \rightarrow \mathbb{N}^r$  is  $f$ -limited if:

$$\max C(x) \leq \max\{1, f(\max x)\}.$$

In this definition we have excluded the possibility of  $\max C(x)$  being limited by 0 to prevent the theorem from being trivially true.

**93 Theorem** (finite adjacent Ramsey with parameter,  $\text{AR}_f$ ). *For every  $k, r$  there exists  $R$  such that for every  $f$ -limited function  $C: R^k \rightarrow \mathbb{N}^r$  there are  $x_1 < \dots < x_{k+1} < R$  with  $C(x_1, \dots, x_k) \leq C(x_2, \dots, x_{k+1})$ .*

We denote these theorems for fixed  $k$  with  $\text{AR}^k$  and  $\text{AR}_f^k$ . They are proved using a standard compactness argument involving König's lemma and the infinite version of the theorem. In the remainder of this chapter we will prove:

**94 Theorem.**  $\text{PA} \not\vdash \text{AR}_{\text{id}}$  and  $\text{IS}_k \not\vdash \text{AR}_{\text{id}}^k$ .

**95 Theorem.** For  $f_\alpha(i) = \log^{H_\alpha^{-1}(i)}(i)$  we have:  $\text{PA} \vdash \text{AR}_{f_\alpha}$  if and only if  $\alpha < \epsilon_0$ .

**96 Theorem.** Given  $k$ , take  $f_\alpha(i) = {}^{H_\alpha^{-1}(i)}\sqrt{\log^k(i)}$ , so  $\text{IS}_{k+1} \vdash \text{AR}_{f_\alpha}^{k+1}$  if and only if  $\alpha < \omega_{k+2}$ .

For the original papers on Ramsey-like independent theorems we refer the reader to [35] for the Paris–Harrington principle, [23] for the Kanamori–McAloon principle. The phase transition results can be found in [45] for the Paris–Harrington theorem, in [8] for the Kanamori–McAloon theorem. For an easy proof of independence of Paris–Harrington and transition results of PH and KM see Chapter 7.

**97 Definition.**  $\text{AR}_f(k, r)$  is the least  $R$  from theorem 93.

We will use terminology from Ramsey theory in a sloppy, i.e. informal way and call multivariate functions ‘colourings’. A colouring  $C: \mathbb{N}^k \rightarrow \mathbb{N}^r$  such that there exist no  $x_1 < \dots < x_{k+1}$  with  $C(x_1, \dots, x_k) \leq C(x_2, \dots, x_{k+1})$  is called bad.

## 6.2 Lower bounds for unlimited dimension

We start with the case of AR. We will later adapt the proofs for the limited dimension cases.

### 6.2.1 The case of the identity function

We first prove the following with respect to  $\text{AR}_f$ .

**98 Theorem.**  $\text{PA} \not\vdash \text{AR}_{\text{id}}$ .

To demonstrate this we follow a proof from [16]. It has been adapted to show that the  $R$  from the theorem has as lower bound the length of a Paris and Kirby Hydra battle, instead of demonstrating that the infinite adjacent Ramsey implies well ordering of  $\varepsilon_0$ . This is sufficient to prove unprovability because the length of such battles starting with  $\alpha$  have lower bound  $H_\alpha(0)$  (See [10]). The author would like to thank Harvey Friedman for his kind permission to copy major parts of his proof.

**99 Definition.** Given  $\alpha = \omega^{\alpha_1} \cdot a_1 + \dots + \omega^{\alpha_n} \cdot a_n$  and  $\beta = \omega^{\beta_1} \cdot b_1 + \dots + \omega^{\beta_m} \cdot b_m$ , with  $\alpha_1 > \dots > \alpha_n$  and  $\beta_1 > \dots > \beta_m$ :

1. The maximal position  $\text{MP}(\alpha)$  is  $\max\{n, \text{MP}(\alpha_i) : 1 \leq i \leq n\}$ .
2. The comparison position  $\text{CP}(\alpha, \beta)$  is the smallest  $i$  such that  $\omega^{\alpha_i} \cdot a_i \neq \omega^{\beta_i} \cdot b_i$  if such an  $i$  exists, zero otherwise.
3. The maximal coefficient  $\text{MC}(\alpha)$  is  $\max\{a_i, \text{MC}(\alpha_i) : 1 \leq i \leq n\}$ .
4. The comparison coefficient  $\text{CC}(\alpha, \beta)$  is  $a_{\text{CP}(\alpha, \beta)}$ , where  $a_0 = 0$ .
5. The comparison exponent  $\text{CE}(\alpha, \beta)$  is  $\alpha_{\text{CP}(\alpha, \beta)}$ , where  $\alpha_0 = 0$ .

**100 Lemma.** We have:

1.  $\text{CP}(\alpha, \beta) \leq \text{MP}(\alpha)$ .
2.  $\text{CC}(\alpha, \beta) \leq \text{MC}(\alpha)$ .
3.  $\text{MP}(\alpha_i) \leq \text{MP}(\alpha)$  and  $\text{MC}(\alpha_i) \leq \text{MC}(\alpha)$ .
4.  $\text{CP}(\alpha, \beta) \leq \text{CP}(\beta, \gamma) \wedge \text{CE}(\alpha, \beta) \leq \text{CE}(\beta, \gamma) \wedge \text{CC}(\alpha, \beta) \leq \text{CC}(\beta, \gamma) \Rightarrow \alpha \leq \beta$ .

*Proof:* The first three properties follow directly from the definitions. For the fourth one let:

$$\begin{aligned} \alpha &= \omega^{\alpha_1} \cdot a_1 + \dots + \omega^{\alpha_n} \cdot a_n, \\ \beta &= \omega^{\beta_1} \cdot b_1 + \dots + \omega^{\beta_m} \cdot b_m, \\ \gamma &= \omega^{\gamma_1} \cdot c_1 + \dots + \omega^{\gamma_p} \cdot c_p. \end{aligned}$$

Take  $i = \text{CP}(\alpha, \beta)$  and  $j = \text{CP}(\beta, \gamma)$ . If  $i = 0$  then  $\alpha = \beta$ . If  $i > 0$  and  $i = j$  then  $\alpha_i \leq \beta_i$  and  $a_i \leq b_i$  and  $\alpha_l = \beta_l$ ,  $a_l = b_l$  for  $l < i$ , so  $\alpha \leq \beta$ . If  $i > 0$  and  $i < j$  then  $\alpha_i \leq \beta_j$ , hence  $\alpha_i < \beta_i$  (if  $\alpha_i \geq \beta_i$  then  $\alpha_i \geq \beta_i > \beta_j$ , which is a contradiction), so  $\alpha < \beta$ .

□

**101 Definition.** A Hydra battle is a sequence  $h_0, \dots, h_l$  of ordinals starting with  $h_0 = \omega_k$  such that  $h_{i+1} = h_i[i + 1]$  and  $h_{i+1} < h_i$  for all  $i < l$ .

**102 Lemma.** For every Hydra battle and  $0 < i \leq l$  we have  $\text{MC}(h_i) \leq i$  and  $\text{MP}(h_i) \leq i$ .

*Proof:* Induction on  $i$ .

□

**103 Definition.** We define  $F_k: \omega_k^k \rightarrow \mathbb{N}^{2k-1}$  by recursion on  $k$ .

1.  $F_1(i) = i$ .
2.  $F_{k+1}(\alpha_1, \dots, \alpha_{k+1}) = (\text{CP}(\alpha_1, \alpha_2), \text{CC}(\alpha_1, \alpha_2), F_k(\text{CE}(\alpha_1, \alpha_2), \dots, \text{CE}(\alpha_k, \alpha_{k+1})))$ .

**104 Lemma.**  $\max F_k(\alpha_1, \dots, \alpha_k) \leq \max\{\text{MC}(\alpha_i), \text{MP}(\alpha_i)\}$ .

*Proof:* Induction on  $k$ , using the first three items of lemma 100.

**105 Lemma.**  $F_k(\alpha_1, \dots, \alpha_k) \leq F_k(\alpha_2, \dots, \alpha_{k+1}) \Rightarrow \alpha_1 \leq \alpha_2$ .

*Proof:* Induction on  $k$ , using lemma 100 item 4.

□

**106 Theorem.**  $\text{AR}_{\text{id}}(k + 1, 2k + 1)$  is larger than the length of a Hydra battle starting with  $\omega_k$ .

*Proof:* Given such a Hydra battle  $h_0, \dots, h_l$ , define the limited (thanks to lemma 102 and lemma 104) colouring  $C: l^{k+1} \rightarrow \mathbb{N}^{2k+1}$ :

$$C: (x_1, \dots, x_{k+1}) \mapsto F_{k+1}(h_{x_1}, \dots, h_{x_{k+1}}).$$

If  $x_1 < \dots < x_{k+2}$  then  $h_{x_1} > h_{x_2}$ , hence  $C(x_1, \dots, x_{k+1}) \not\leq C(x_2, \dots, x_{k+2})$  by the previous lemma.

□

This also finishes the proof of theorem 98. We can get the original proof from [16] by defining colourings for all  $k$  on  $\mathbb{N}^k$  using  $F_k$  for any infinite decreasing sequence in  $\omega_k$ . The infinite adjacent Ramsey theorem will imply that those sequences are not strictly decreasing, thus that  $\varepsilon_0$  is well-ordered.

### 6.2.2 The case of the iterated logarithm

We modify the colouring, using estimates of  $\text{AR}_c$ . Here one can see how a direct relation emerges between estimates on the version of the adjacent Ramsey theorem with constant function (Provable in PA, or even  $\text{IS}_1$ ) and the transition for the unprovable version. We show that  $\text{AR}_c(k, r)$  cannot be bounded by a tower of exponentials of fixed height. This will imply that the unprovability of  $\text{AR}_f$  is preserved for parameter values equal to the inverse of those towers.

The basic construction is simple, given a bad colouring  $C$  for identity we will take new colouring that looks like:

$$\tilde{C}(x) = (w(x), D_{\log^n(x_k)}(x), C(\log^n(x_1), \dots, \log^n(x_k)))$$

Where  $w$  will be such that for  $x_1 < \dots < x_{k+1}$  if  $w(x_1, \dots, x_k) \leq w(x_2, \dots, x_{k+1})$  then either  $\log^n x_1 < \dots < \log^n x_{k+1}$  or  $\log^n x_k = \log^n x_{k+1}$ .  $D_i$  will be such that it is a bad colouring of  $2_n(i+1)^k$ . This construction implies that, for  $x_1 < \dots < x_{k+1}$ , the property  $\tilde{C}(x_1, \dots, x_k) \not\leq \tilde{C}(x_2, \dots, x_{k+1})$  is inherited from its components.

For this construction we will need to have the appropriate bad colourings  $D_i$ , so we start with:

**107 Lemma.** *Suppose we have a colouring  $C: R^k \rightarrow c^r$  such that  $C(x_1, \dots, x_k) \neq C(x_2, \dots, x_{k+1})$  for all  $x_1 < \dots < x_{k+1} < R$ , then there exists a colouring  $D: R^k \rightarrow c^{2r}$  such that  $D(x_1, \dots, x_k) \not\leq D(x_2, \dots, x_{k+1})$  for all  $x_1 < \dots < x_{k+1} < R$ .*

*Proof:* Take  $D(x) = (C(x), c - C(x) - 1)$ .

□

**108 Lemma.** *For any  $k \geq 0, c > 1$  there exists a colouring  $C: 2_k(c+1)^{k+1} \rightarrow c^{64^{k \cdot 2}}$  such that for any  $x_1 < \dots < x_{k+1} < 2_k(c+1)$  we have  $C(x_1, \dots, x_k) \neq C(x_2, \dots, x_{k+1})$ .*

*Proof:* Induction on  $k$ : First  $C(0) = (c - 1, c - 1)$ ,  $C(1) = (c - 2, c - 1)$ ,  $C(x) = (c - 2, c - x + 1)$  is such a colouring for  $k = 0$ .

For the induction step suppose there is such a colouring  $D$  for  $k$ . The following construction and proof are taken from the proof of lemma 1.9 in [16]. We define some functions  $h_1, \dots, h_7$ :

$$h_1(x_1, \dots, x_{k+1}) = \begin{cases} 0 & \text{if } x_1 = 0, \\ 1 & \text{otherwise.} \end{cases}$$

$h_2(x_1, \dots, x_{k+1}) = 0$  if  $x_1 = 0$ , otherwise it is the parity of the greatest  $i$  such that  $\log x_1 = \dots = \log x_i$ .

$h_3(x_1, \dots, x_{k+1}) = 0$  if  $x_1 = 0$ , otherwise it is the parity of the greatest  $i$  such that  $\log x_1 < \dots < \log x_i$ .

Let  $l(x, y)$  be the first digit  $x$  and  $y$  differ at in base two (from the left, assuming that they have the same leading digit).

$h_4(x_1, \dots, x_{k+1}) =$  the parity of the greatest  $i$  such that  $l(x_1, x_2) = \dots = l(x_i, x_{i+1})$  assuming that  $\log x_1 = \dots = \log x_{k+1}$ . It is 0 otherwise.

$h_5(x_1, \dots, x_{k+1}) =$  the parity of the greatest  $i$  such that  $l(x_1, x_2) < \dots < l(x_i, x_{i+1})$  assuming that  $\log x_1 = \dots = \log x_{k+1}$ . It is 0 otherwise.

$h_6(x_1, \dots, x_{k+1}) =$  the parity of the greatest  $i$  such that  $l(x_1, x_2) > \dots > l(x_i, x_{i+1})$  assuming that  $\log x_1 = \dots = \log x_{k+1}$ . It is 0 otherwise.

$$h_7(\bar{x}) = \begin{cases} D(\log x_1, \dots, \log x_k) & \text{if } \log x_1 < \dots < \log x_k, \\ D(l(x_1, x_2), \dots, l(x_k, x_{k+1})) & \text{if } l(x_1, x_2) < \dots < l(x_k, x_{k+1}), \\ D(l(x_k, x_{k+1}), \dots, l(x_1, x_2)) & \text{if } l(x_1, x_2) > \dots > l(x_k, x_{k+1}), \\ 0 & \text{otherwise.} \end{cases}$$

We combine  $h_1, \dots, h_7$  into a single function  $C$  with range  $2^6 \times c^{64^{k+1} \cdot 2} \subseteq c^{64^{k+1} \cdot 2}$  and domain  $(2^{2k(c+1)})^{k+2}$ .

Suppose, for a contradiction, that  $x_1 < \dots < x_{k+2}$  are such that  $C(x_1, \dots, x_{k+1}) = C(x_2, \dots, x_{k+2})$ . So  $x_2 > 0$  hence  $C(x_1, \dots, x_{k+1}) = 1$  thus  $x_1 > 0$ . Fur-



thermore  $h_2(x_1, \dots, x_{k+1}), h_3(x_1, \dots, x_{k+1}) \in \{1, k+1\}$ , because otherwise  $h_2(x_1, \dots, x_{k+1}) \neq h_2(x_2, \dots, x_{k+2})$  (same for  $h_3$ , the parity changes if the variables shift). Using the same argument  $h_4(x_1, \dots, x_{k+1}), h_5(x_1, \dots, x_{k+1}), h_6(x_1, \dots, x_{k+1}) \in \{1, k\}$ . So, by the properties of  $h_2, h_3$  we have two possibilities:

$$\begin{aligned} \log x_1 &= \dots = \log x_{k+2} \text{ or} \\ \log x_1 &< \dots < \log x_{k+2}. \end{aligned}$$

In the latter case  $C(x_1, \dots, x_{k+1}) \neq C(x_2, \dots, x_{k+2})$  is inherited from  $D$  (by definition of  $h_7$ ), so we have the first one. In this case we have, by the properties of  $h_4, h_5, h_6$ :

$$\begin{aligned} l(x_1, x_2) &= \dots = l(x_{k+1}, x_{k+2}), \\ l(x_1, x_2) &> \dots > l(x_{k+1}, x_{k+2}), \text{ or} \\ l(x_1, x_2) &< \dots < l(x_{k+1}, x_{k+2}). \end{aligned}$$

In the latter two cases  $C(x_1, \dots, x_{k+1}) \neq C(x_2, \dots, x_{k+2})$  is inherited from  $D$  again, so the first case remains, which is impossible because in that case either  $x_3 < x_2$  or  $x_2 < x_1$  (the binary lengths of  $x_1, x_2, x_3$  are equal because  $\log x_1 = \log x_2$ ), obtaining our contradiction and finishing the proof.

□

**109 Theorem.**  $\text{PA} \not\vdash \text{AR}_{\log^n}$ .

It is sufficient to show for  $k > n$  that we can modify the bad id-limited colouring  $C: R^{k+1} \rightarrow \mathbb{N}^r$  into a bad  $\log^n$ -limited colouring  $\tilde{C}: R^{k+1} \rightarrow \mathbb{N}^{64^k \cdot 4 + k + r}$ . Combine lemmas 107 and 108 into functions  $D_c: 2_k(c+1)^{k+1} \rightarrow c^{64^k \cdot 4}$ . Take  $w(i) = (j_1, \dots, j_k)$ , where  $j_i = 0$  and the other  $j$ 's are 1. Define the new colouring:

$$\tilde{C}(x_1, \dots, x_k) = (w(i), D_{\log^n(x_k)}(x), C(\log^n(x_1), \dots, \log^n(x_k))),$$

if  $1 < i \leq k$  is the maximum such that  $\log^n(x_{i-1}) = \log^n(x_i)$  and

$$\tilde{C}(x_1, \dots, x_k) = (w(1), 0, \dots, 0, C(\log^n(x_1), \dots, \log^n(x_k))),$$

if such an  $i$  does not exist. If  $x_1 < \dots < x_{k+1}$  we have two cases:

1. If  $\log^n(x_1) < \dots < \log^n(x_{k+1})$ , then  $\tilde{C}(x_1, \dots, x_k) \not\leq \tilde{C}(x_2, \dots, x_{k+1})$  is inherited from  $C$ .

2. There exists  $i$  such that  $\log^n(x_{i-1}) = \log^n(x_i) < \log^n(x_{i+1}) < \dots < \log^n(x_{k+1})$ . If  $i = k + 1$  and  $\log^n(x_{k-1}) = \log^n(x_k)$  then  $\tilde{C}(x_1, \dots, x_k) \not\leq \tilde{C}(x_2, \dots, x_{k+1})$  is inherited from  $D$ . If  $i < k + 1$  then that property is guaranteed by  $w$ .

□

**110 Theorem.** For  $f(i) = \log^{H_{\varepsilon_0}^{-1}(i)}(i)$  we have  $\text{PA} \not\vdash \text{AR}_f$ .

*Proof:* For  $k > n$  we have the estimate:

$$\text{AR}_{\log^n}(k+1, 64^k \cdot 4 + 3k + 1) \geq H_{\omega_k}(0).$$

We claim that for  $k > 2n$ :

$$R = \text{AR}_f(k+1, 64^k \cdot 4 + 3k + 1) \geq H_{\varepsilon_0}(n).$$

Suppose  $R < H_{\varepsilon_0}(n)$ , then for  $i \leq R$  we have  $H_{\varepsilon_0}^{-1}(i) \leq n$ . Hence

$$\begin{aligned} R &\geq \text{AR}_{\log^n}(k+1, 64^k \cdot 4 + 3k + 1) \\ &\geq H_{\omega_k}(0) \\ &\geq H_{\omega_n}(n) \end{aligned}$$

which contradicts the assumption, so  $R \geq H_{\varepsilon_0}(n)$ .

□

### 6.3 Lower bounds for limited dimension

We modify these proofs to classify the transitions for limited dimensions in fragments of PA. Unlike in the case of the Paris–Harrington or in the case of the Kanamori–McAloon theorems where dimension  $k+1$  is required,  $\text{AR}_{\text{id}}^k$  is already not provable in  $\text{I}\Sigma_k$ . This difference in dimensions can be explained by the fact that there is ‘an extra dimension’ hidden in the unlimited number of dimensions of the range of the colourings. So, for example, parametrised adjacent Ramsey with  $k = 1$  and  $f = \text{id}$  is equivalent to Dickson’s lemma. This observation is the main idea for providing the proper starting point for  $F_1$  and for improving the lower bound estimates for the theorem with constant parameter. We start by improving theorem 106. We will extend our existing definition of Hydra battles.

**111 Definition.** A Hydra battle is a sequence  $h_0, \dots, h_L$  of ordinals starting with  $h_0 = \omega_k(l)$  such that  $h_{i+1} = h_i[i+1]$  and  $h_{i+1} < h_i$  for all  $i < L$ .

Using the notations from this definition.

**112 Lemma.** For every Hydra battle and  $i \leq L$  we have  $\text{MC}(h_i) \leq l + i + 1$  and  $\text{MP}(h_i) \leq l + i + 1$ .

*Proof:* By induction on  $i$ .

□

We use these results to show unprovability in the fragments of PA:

**113 Theorem.**  $\text{AR}_{\text{id}}(k, 3k + (l+1)^k + l + 2)$  is larger than the length of a Hydra battle starting with  $\omega_k(l)$ .

*Proof:* We first define a modified version of the  $F$  from definition 103 :

**114 Definition.** We define  $F_k^l: \omega_k(l+1)^k \rightarrow \mathbb{N}^{2k+l}$  by recursion on  $k$ .

1. Given  $\alpha = \omega^l \cdot n_l + \dots + \omega^0 \cdot n_0$ , take  $F_1^l(\alpha) = (n_l, \dots, n_0)$ .
2.  $F_{k+1}^l(\alpha_1, \dots, \alpha_{k+1}) = (\text{CP}(\alpha_1, \alpha_2), \text{CC}(\alpha_1, \alpha_2), F_k^l(\text{CE}(\alpha_1, \alpha_2), \dots, \text{CE}(\alpha_k, \alpha_{k+1})))$ .

Given the length of such a Hydra battle and  $C$  as defined in the proof of theorem 98, taking into account the modified  $F_k^l$ .  $C$  is not an id-limited colouring! Fortunately we can still use lemma 112. Take  $V(i) = (j_1, \dots, j_{(l+1)^k})$ , where  $j_i = 0$  and the other  $j$ 's are 1 and enumerate the elements of  $\{0, \dots, l\}^k$  with  $r_1, \dots, r_{(l+1)^k}$ . Define the following id-limited colouring:

$$D(r_i) = (0, V(i), 0),$$

if there exists  $j$  with  $x_j > l$  and  $x_{j-1} \leq l$ , then

$$D(x) = (w(j), 0, 0),$$

where  $j$  is the minimum such  $j$  and  $w$  is as in the proof of 109. If all  $x_j > l$ , then:

$$D(x) = (w(1), 0, C(x - l - 1)).$$

Suppose  $x_1 < \dots < x_{k+2}$ , we have three cases:

1. If  $x_{k+2} \leq l$ , then  $D(x_1, \dots, x_{k+1}) \not\preceq D(x_2, \dots, x_{k+2})$  is inherited from  $V$ .
2. If  $x_1 > l$ , then  $D(x_1, \dots, x_{k+1}) \not\preceq D(x_2, \dots, x_{k+2})$  is inherited from  $C$ .

3. There exists  $j$  with  $x_j > l$  and  $x_{j-1} \leq l$ , if  $j = k + 2$  then  $D(x_1, \dots, x_{k+1})$  has some coordinates equal to 1 (from the definition of  $V$ ) where in  $D(x_2, \dots, x_{k+2})$  they are equal to 0. In the other case

$$D(x_1, \dots, x_{k+1}) \not\leq D(x_2, \dots, x_{k+2})$$

is ensured by  $w$ .

□

**115 Theorem.** For any  $c > 1$  we have:  $\text{IS}_{k+1} \not\vdash \text{AR}_{\sqrt[c]{\log^k}}^{k+1}$ .

*Proof:* Take  $f(i) = \sqrt[c]{\log^k i}$ . We first improve estimates of lower bounds of  $\text{AR}_{\text{id}}$ .

**116 Lemma.** For any  $c, d, k \geq 0$  there exists a colouring

$$C: 2_k((d+1)^c)^{k+1} \rightarrow d^{64^k \cdot 2 \cdot c}$$

such that for any  $x_1 < \dots < x_{k+1} < 2_k((d+1)^c)^{k+1}$  we have:

$$C(x_1, \dots, x_k) \neq C(x_2, \dots, x_{k+1}).$$

*Proof:* Induction on  $k$ : Enumerate the elements of  $(d+1)^{2 \cdot c}$  with  $r_0, r_1, \dots$ . So  $C(i) = r(i)$  is such a colouring for  $k = 0$ . For the induction step follow the proof of lemma 108.

□

We combine lemmas 116 and 107 with our colouring for the identity as in the unlimited dimension case.  $C$  is the colouring for the identity,

$$D_d: 2_k((d+1)^c)^{k+1} \rightarrow d^{64^k \cdot 4 \cdot c},$$

the colouring obtained from combining the two lemmas. The new colouring is:

$$\tilde{C}(x_1, \dots, x_{k+1}) = (w(i), D_{f(x_{k+1})}(x), C(f(x_1), \dots, f(x_{k+1}))),$$

if  $1 < i \leq k+1$  is the maximum such that  $f(x_{i-1}) = f(x_i)$  and

$$\tilde{C}(x_1, \dots, x_{k+1}) = (w(1), 0, C(f(x_1), \dots, f(x_{k+1}))),$$

if such an  $i$  does not exist. Suppose  $x_1 < \dots < x_{k+2}$ , then we have the following cases:

1. If  $f(x_1) < \dots < f(x_{k+2})$ , then  $\tilde{C}(x_1, \dots, x_{k+1}) \not\leq \tilde{C}(x_2, \dots, x_{k+2})$  is inherited from  $C$ .

2. If there exists  $i$  such that  $f(x_{i-1}) = f(x_i) < f(x_{i+1}) < \dots < f(x_{k+2})$ . If  $i = k + 2$  and  $f(x_k) = f(x_{k+1})$ , then  $\tilde{C}(x_1, \dots, x_{k+1}) \not\leq \tilde{C}(x_2, \dots, x_{k+2})$  is inherited from  $D$ . If  $i < k + 2$  then that property is guaranteed by  $w$ .

This finishes the proof of theorem 115.

□

**117 Theorem.** For  $f(i) = {}^{H_{\omega_{k+2}}^{-1}}\sqrt[i]{\log^k(i)}$  we have  $\text{IS}_{k+1} \not\vdash \text{AR}_f^{k+1}$ .

*Proof:* Repeating the proof for the unlimited dimension case, we have the estimate:

$$\text{AR}_{\sqrt[k]{\log^k}}(k+1, 4k + (l+1)^k + l + 2 + 64^k \cdot 4 \cdot c) \geq H_{\omega_{k+1}(l)}(0).$$

We claim that:

$$R = \text{AR}_f(k+1, 4k + (l+1)^k + l + 2 + 64^k \cdot 4 \cdot 2l) \geq H_{\omega_{k+2}(l)}.$$

Suppose  $R < H_{\omega_{k+1}(l)}$ , then for  $i \leq R$  we have  $H_{\omega_{k+1}}^{-1}(i) \leq l$ . Hence

$$\begin{aligned} R &\geq \text{AR}_{\sqrt[k]{\log^k}}(k+1, 4k + (l+1)^k + l + 2 + 64^k \cdot 4 \cdot 2l) \\ &\geq H_{\omega_{k+1}(2l)}(0) \\ &\geq H_{\omega_{k+2}(l)}. \end{aligned}$$

which contradicts the assumption, so  $R \geq H_{\omega_{k+2}(k)}$ .

□

## 6.4 Upper bounds

### 6.4.1 Upper bounds for unlimited dimension

We use known bounds on ordinary Ramsey numbers which state that Ramsey numbers for dimension  $d$  are bounded by a tower of exponentials of height  $d - 1$ .

**118 Theorem.** If  $\alpha < \varepsilon_0$  and  $f_\alpha = \log^{H_\alpha^{-1}(i)}(i)$ , then  $\text{PA} \vdash \text{AR}_{f_\alpha}$

*Proof:* Assume without loss of generality that  $r > k$ . We show that:

$$\text{AR}_{f_\alpha}(k, r) \leq R = 2_k(H_\alpha(k + r + 2)^{r+2}).$$

For  $f_\alpha$ -limited colouring  $C$  of  $R$  we know that the max of the range is bounded by:

$$\log^{r+2} H_\alpha(r+2)^{r+2} \leq H_\alpha(k+r+2).$$

Hence the number of colours is bounded by  $H_\alpha(k+r+2)^{r+1}$ , so by Erdős-Rado bounds on Ramsey numbers (See theorem 1 from [13]) the colouring obtained by restricting  $C$  to  $k$ -element subsets of  $R$  contains a homogeneous set of size  $k+1$ , enumerate this set to obtain the required  $x_1 < \dots < x_{k+1}$ .

□

**119 Corollary.**  $\text{PA} \vdash \text{AR}_{\log^*}$ .

### 6.4.2 Upper bounds for limited dimension

**120 Theorem.** *If  $\alpha < \omega_{k+2}$  and  $f_\alpha(i) = {}^{H_\alpha^{-1}(i)}\sqrt{\log^k(i)}$ , then  $\text{IS}_{k+1} \vdash \text{AR}_{f_\alpha}^{k+1}$ .*

*Proof:* Assume without loss of generality that  $r > k$ . We show that:

$$\text{AR}_{f_\alpha}(k, r) \leq R = 2_k(H_\alpha(r+2)^{r+2}).$$

For  $f_\alpha$ -limited colouring  $C$  of  $R$  we know that the range is bounded by:

$${}^{r+2}\sqrt{H_\alpha(r+2)^{r+2}} \leq H_\alpha(r+2).$$

Hence the number of colours is bounded by  $H_\alpha(r+2)^{r+1}$ , so by Erdős-Rado bounds on Ramsey numbers (See theorem 1 from [13]) the colouring obtained by restricting  $C$  to  $k+1$ -element subsets of  $R$  contains a homogeneous set of size  $k+2$ , enumerate this set to obtain the required  $x_1 < \dots < x_{k+2}$ .

□

**121 Corollary.**  $\text{IS}_{k+1} \vdash \text{AR}_{\log^{k+1}}^{k+1}$ .

## 6.5 Other versions of the adjacent Ramsey theorem

In this chapter we have only examined one of Friedman's variants of the adjacent Ramsey principle. In [16] he examines eight versions of this principle from the viewpoint of reverse mathematics. The following theorem lists his results and is a worthy research topic in its own right.

**122 Theorem (Friedman).** *The following are equivalent over  $\text{RCA}_0$ :*

- A For all  $f: \mathbb{N}^k \rightarrow \mathbb{N}^2$  there exist distinct  $x_1, \dots, x_{k+1}$  such that  $f(x_1, \dots, x_k) \leq f(x_2, \dots, x_{k+1})$ .*
  - B For all  $f: \mathbb{N}^k \rightarrow \mathbb{N}$  there exist distinct  $x_1, \dots, x_{k+2}$  such that  $f(x_1, \dots, x_k) \leq f(x_2, \dots, x_{k+1}) \leq f(x_3, \dots, x_{k+2})$ .*
  - C For all  $f: \mathbb{N}^k \rightarrow \mathbb{N}$  there exist distinct  $x_1, \dots, x_{k+1}$  such that  $f(x_2, \dots, x_{k+1}) - f(x_1, \dots, x_k) \in 2\mathbb{N}$ .*
  - D For all  $f: \mathbb{N}^k \rightarrow \mathbb{N}^r$  there exist  $x_1 < \dots < x_{k+1}$  such that  $f(x_1, \dots, x_k) \leq f(x_2, \dots, x_{k+1})$ .*
  - E For all  $t \geq 1$  and  $f: \mathbb{N}^k \rightarrow \mathbb{N}^r$  there exist  $x_1 < \dots < x_{k+t-1}$  such that  $f(x_1, \dots, x_k) \leq \dots \leq f(x_t, \dots, x_{k+t-1})$ .*
  - F For all  $t \geq 1$  and  $f: \mathbb{N}^k \rightarrow \mathbb{N}^r$  there exist  $x_1 < \dots < x_{k+1}$  such that  $f(x_2, \dots, x_{k+1}) - f(x_1, \dots, x_k) \in t\mathbb{N}^r$ .*
  - G For all  $f: \mathbb{N}^k \rightarrow 2, g: \mathbb{N}^k \rightarrow \mathbb{N}$  there exist distinct  $x_1, \dots, x_{k+1}$  such that  $f(x_1, \dots, x_k) = f(x_2, \dots, x_{k+1})$  and  $g(x_1, \dots, x_k) \leq g(x_2, \dots, x_{k+1})$ .*
  - H For all  $f: \mathbb{N}^k \rightarrow c, g: \mathbb{N}^k \rightarrow \mathbb{N}$  there exist distinct  $x_1, \dots, x_{k+1}$  such that  $f(x_1, \dots, x_k) = f(x_2, \dots, x_{k+1})$  and  $g(x_1, \dots, x_k) \leq g(x_2, \dots, x_{k+1})$ .*
- $\varepsilon_0$  is well-ordered.

The finite versions of these statements using limited functions are also equivalent. It will be of great interest to examine phase transitions for these theorems and the versions with restricted dimensions. In theorems A,B,C,G,H the dimension of the range of the functions is restricted, so we expect in those cases the restricted variants with dimension  $k+1$  to be unprovable in  $\text{I}\Sigma_k$ , and for the variants of E,F the same, but as in the case of theorem D, with dimension  $k$ .





## Chapter 7

# Ramsey type independence

The techniques from the previous chapter seem eminently suitable to treat the two other Ramsey-like unprovable theorems. We will reprove existing results on the independence of the Paris–Harrington theorem and phase transitions for that theorem and the Kanamori–McAloon theorem. The goal in this chapter is a unified treatment of these theorems.

### 7.1 A short proof of the Paris–Harrington result

Recall the Paris–Harrington theorem:

**123 Theorem (PH).** *For every  $d, c, m$  there exists an  $R$  such that for every colouring  $C: [m, R]^d \rightarrow c$  there exists an  $H \subseteq [m, R]$  of size  $\min H$  for which  $C$  limited to  $[H]^d$  is constant.*

**124 Definition.** *We denote the least such  $R$  with  $\text{PH}^d(m, c)$ .*

We will re-prove independence of PA and independence of this theorem of  $\text{IS}_d$  for fixed dimension  $d + 1$  using the colourings from Adjacent Ramsey, providing the easiest proof using function hierarchies known to the author.

Recall the definition of  $F_k^l$ :

**125 Definition.** *We define  $F_k^l: \omega_k(l + 1)^k \rightarrow \mathbb{N}^{2k+l}$  by recursion on  $k$ .*

1. *Given  $\alpha = \omega^l \cdot n_l + \dots + \omega^0 \cdot n_0$ , take  $F_1^l(\alpha) = (n_l, \dots, n_0)$ .*
2.  *$F_{k+1}^l(\alpha_1, \dots, \alpha_{k+1}) = (\text{CP}(\alpha_1, \alpha_2), \text{CC}(\alpha_1, \alpha_2), F_k^l(\text{CE}(\alpha_1, \alpha_2), \dots, \text{CE}(\alpha_k, \alpha_{k+1})))$ .*

By lemma 104 this function has the following property:

$$\max F_k^l(\alpha_1, \dots, \alpha_k) \leq \max\{\text{MC}(\alpha_i), \text{MP}(\alpha_i)\}.$$

Upon closer inspection, we observe the following behaviour:

**126 Lemma.**  $\max F_k^l(\alpha_1, \dots, \alpha_k) \leq \max\{\text{MC}(\alpha_1), \text{MP}(\alpha_1)\}.$

This is proved with induction on  $k$ .

**127 Theorem.**  $\text{PH}^{k+1}(2l + k + 2, 2k + l + 1) \geq H_{\omega_k(l)}(0)$  for  $l \geq 2$ .

*Proof:* Let  $R$  be the length of a Hydra battle starting with  $\omega_k(l)$ . We define a colouring  $C: [2l + k + 2, R]^{k+1} \rightarrow 2k + l + 1$  as follows:

$$C(x_1, \dots, x_{k+1}) = \begin{cases} 0 & \text{if } F_k^l(h_{x_1-2l-k-2}, \dots, h_{x_k-2l-k-2}) \leq \\ & F_k^l(h_{x_2-2l-k-2}, \dots, h_{x_{k+1}-2l-k-2}), \\ i & \text{otherwise,} \end{cases}$$

where  $i$  is the least such that:

$$(F_k^l(h_{x_1-2l-k-2}, \dots, h_{x_k-2l-k-2}))_i > (F_k^l(h_{x_2-2l-k+2}, \dots, h_{x_{k+1}-2l-k-2}))_i.$$

Observe that  $F_k^l(h_{x_1-2l-k-2}, \dots, h_{x_k-2l-k-2})_i \leq x_1 - l - k - 1$  (this is a consequence of lemmas 126 and 112). Suppose we have an  $H \subseteq [2l + k + 2, R]$  which is homogeneous for  $C$ , then  $C|_{[H]^{k+1}} \neq 0$  thanks to lemma 105, which is also true for  $F_k^l$ . Hence, due to the definition of  $C$ , we can obtain a strictly descending sequence starting with  $a \leq x_1 - l - k - 1$  of size  $|H| - k - 2$ . So  $H$  must have size strictly less than  $\min H$ . □

**128 Corollary.**  $\text{PA} \not\vdash \text{PH}$  and  $\text{IS}_d \not\vdash \text{PH}^{d+1}$ .

## 7.2 Phase transitions for PH and KM

For both the Paris–Harrington and Kanamori–McAloon theorems the principle of showing the unprovability result for the parameter functions will be the same. Take a counterexample colouring for the identity function and modify this in the following manner:

If  $f(x_1) < \dots < f(x_d)$ , take the original value for  $f(x_1), \dots, f(x_d)$ .

If  $f(x_1) = \dots = f(x_d)$ , take the value of a colouring  $D_{f(x_1)}$  which is a bad colouring for the theorem with the parameter equal to the constant function with value  $f(x_1)$ . This limits how slow the function  $f$  is allowed to grow because selecting such colouring will certainly be impossible if the domain of  $D_{f(x_1)}$  is allowed to exceed, for example, the Erdős–Rado bounds for Ramsey numbers from [13].

Ensure that in the other cases there will be no (min-)homogeneity by using an adjacent Ramsey type argument. This argument will be slightly more complicated because the sizes of the homogeneous sets/number of colours depend on the minimal element.

This approach sheds light on the difference in transition results between the two statements: In the case of the Kanamori–McAloon theorem the domains of the functions  $D_i$  vary according to the number of colours allowed, keeping the size constant. For Paris–Harrington they vary according to the allowed size of the homogeneous sets, whilst keeping the number of colours constant. If we examine the corresponding lower bounds for the versions with constant parameter we see that in both cases we have that, roughly speaking, the size of the allowed homogeneous sets sits on top of the exponential tower, whilst the number of colours is one level lower. This ensures that the domains of the functions  $D_i$  as a function of  $i$  for PH will be one exponent higher compared to those of KM, thus allowing for an extra iteration of log for the parameter  $f$ . Note also that, in the KM case this allows one to compensate for roots by increasing the size of min-homogeneous set by one, whilst for PH it is only possible to compensate for division, by taking an exponential (fixed, dependent on the divisor) number of colours.

### 7.2.1 Paris–Harrington

Recall the Paris–Harrington theorem:

**129 Theorem** ( $\text{PH}_f$ ). *For every  $d, r, m$  there exists an  $R$  such that for every colouring  $C: [m, R]^d \rightarrow r$  there exists an  $H \subseteq [m, R]$  of size  $f(\min H)$  for which  $C$  limited to  $[H]^d$  is constant.*

**130 Definition.** *We denote the least such  $R$  with  $\text{PH}_f^d(m, r)$ . We call a colouring  $C: [m, R]^d \rightarrow r$  bad if every  $C$ -homogeneous set has size strictly less than  $f(\min H)$ .*

We will use lower bounds from Ramsey theory from [20], which are attributed to Erdős and Hajnal:

**131 Lemma.** *For every  $d \geq 2$  there exists constant  $a_d$  such that*

$$\text{PH}_{i \cdot a_d}^d(0, r) > 2_{d-2}(r^{i-2})$$

*for all  $r \geq 4$  and  $i \geq 3$ .*

**132 Theorem.**  $\text{PH}_{\frac{\log^d}{c}}^{d+1}(2_d(a_d \cdot c \cdot m), r \cdot (d+2)^2 \cdot 2^{(c+2) \cdot a_d}) \geq \text{PH}_{\text{id}}^{d+1}(m, r)$  for every  $c > 0$  and  $m$  sufficiently large.

*Proof:* We start with a description of the proof. The idea is to take some bad colouring  $D$  for the identity function and convert this to a bad colouring for  $g = \frac{\log^d}{c}$ . The obvious way is to use

$$D(g(x_1), \dots, g(x_{d+1})).$$

Notice that this function is not yet well defined, at the places it is not we define the value to be 0. To ensure that the colouring remains bad we add three coordinates. Two of the coordinates will guarantee that for homogeneous sets they imply either  $g(x_1) = \dots = g(x_{d+1})$  or  $g(x_1) < \dots < g(x_{d+1})$  in a manner similar to the constructions we have seen in the chapter on adjacent Ramsey, the other coordinate will be the bad colouring  $D_{x_1}$  whose existence is guaranteed by  $\text{PH}_{x_1}(0, i)$  for a fixed  $i$ . This results in a bad colouring for the case that  $g(x_1) = \dots = g(x_{d+1})$ . The numbers inside the estimate are chosen in such a manner that the colourings will be well-defined and such that we will be able to use the lower bound estimates from lemma 131.

Take  $f(i) = \frac{\log^d(i)}{a_d \cdot c}$  and colourings  $D: [m, R]^{d+1} \rightarrow r$ ,

$$D_{i \cdot a_d}: [2_{d-1}(2^{(c+1) \cdot a_d \cdot i})]^{d+1} \rightarrow 2^{(c+2) \cdot a_d},$$

where  $D$  is a bad colouring for the identity and  $D_{i \cdot a_d}$  is obtained from lemma 131. Define

$$C: [2_d(a_d \cdot c \cdot m), R]^{d+1} \rightarrow r \times 2^{(c+2) \cdot a_d} \times (d+2)^2$$

as follows:

If  $f(x_1) < \dots < f(x_{d+1})$  then:

$$C(x) = (D(f(x_1), \dots, f(x_{d+1})), 0, 0, d+1).$$

If  $1 < i < d+1$  is maximal such that  $f(x_1) = \dots = f(x_i)$  and  $1 \leq j < d+1$  is maximal such that  $f(x_1) < \dots < f(x_j)$ , then:

$$C(x) = (0, 0, i, j).$$

Otherwise:

$$C(x) = (0, D_{f(x_1) \cdot a_d}(x), d+1, 0).$$

Suppose that  $H$  is homogeneous for  $C$  and of size greater than  $d+2$ . In this case the last two coordinates have value 0 or  $d+1$ . If not then there exist  $x_1 < \dots < x_{d+2}$  with  $i = (C(x_2, \dots, x_{d+2}))_3 = (C(x_1, \dots, x_{d+2}))_3 + 1 = i + 1$ , contradiction (same argument for 4th coordinate). If one of those two is 0 then the other must be  $d+1$ , so either  $f(x_1) = \dots = f(x_{d+1})$  for all  $x_1 < \dots < x_{d+1}$  in  $H$  or  $f(x_1) < \dots < f(x_{d+1})$  for all  $x_1 < \dots < x_{d+1}$  in  $H$ .

By definition of  $C$  this implies that  $H$  is homogeneous for  $D$  or  $D_{f(\min H) \cdot a_d}$ , in both cases the size of  $H$  is strictly less than  $\frac{\log^d(\min H)}{c}$ .

□

**133 Corollary.**  $\text{IS}_d \not\vdash \text{PH}_{\frac{\log^d}{c}}^{d+1}$  for every  $c > 0$ .

**134 Corollary.**  $\text{PA} \not\vdash \text{PH}_{\log^n}$  for every  $n$ .

We can sharpen this result in the following manner.

**135 Theorem.**  $\text{IS}_d \not\vdash \text{PH}_f^{d+1}$ , where  $f(i) = \frac{\log^d(i)}{H_{\omega_{d+1}}^{-1}(i)}$ .

*Proof:* Use existing estimates, we already know for  $l > d+2$ :

$$\text{PH}_{\frac{\log^d}{c}}^{d+1}(2_d(a_d \cdot c \cdot 3l), 3l^3 \cdot 2^{(c+2) \cdot a_d}) \geq H_{\omega_d(l)}(0).$$

We claim that  $\text{PH}_f^{d+1}(2_d(a_d \cdot 3(2l)^2), 3(2l)^3 \cdot 2^{(2l+2) \cdot a_d}) \geq H_{\omega_{d+1}}(l)$ . Assume for a contradiction that:

$$R = \text{PH}_f^{d+1}(2_d(a_d \cdot 3(2l)^2), 3(2l)^3 \cdot 2^{(2l+2) \cdot a_d}) < H_{\omega_{d+1}}(l).$$

So  $H_{\omega_{d+1}}^{-1}(i) \geq l$  for  $i < R$ , hence  $\frac{\log^d(i)}{H_{\omega_{d+1}}^{-1}(i)} \leq \frac{\log^d(i)}{l}$  for such  $i$ . So we have the following:

$$\begin{aligned} R &\geq \text{PH}_{\frac{\log^d}{l}}^{d+1}(2_d(a_d \cdot 3(2l)^2), 3(2l)^3 \cdot 2^{(2l+2) \cdot a_d}) \\ &\geq H_{\omega_d(2l)}(0) \\ &\geq H_{\omega_{d+1}}(l). \end{aligned}$$

Where the first inequality follows directly from the definition of  $\text{PH}_f^{d+1}$ . Contradiction!

□

**136 Theorem.**  $\text{PA} \not\vdash \text{PH}_f$ , where  $f(i) = \log^{H_{\varepsilon_0}^{-1}(i)}(i)$ .

*Proof:* We again use existing estimates:

$$\text{PH}_{\log^d}^{d+1}(2_d(a_d \cdot 4 \cdot 3l), 3l^3 \cdot 4^{a_d}) \geq H_{\omega_d(l)}(0).$$

We claim  $\text{PH}_f^{l+1}(2_d(a_d \cdot 4 \cdot 3(2l)), 3(2l)^3 \cdot 4^{a_d}) \geq H_{\varepsilon_0}(l)$ . Assume for contradiction that:

$$R = \text{PH}_f^{l+1}(2_d(a_d \cdot 4 \cdot 3(2l)), 3(2l)^3 \cdot 4^{a_d}) < H_{\varepsilon_0}(l),$$

so  $\log^{H_{\varepsilon_0}^{-1}(i)}(i) \leq l$  for  $i \leq R$ . Hence:

$$\begin{aligned} R &\geq \text{PH}_{\log^l}^{l+1}(2_d(a_d \cdot 4 \cdot 3(2l)), 3(2l)^3 \cdot 4^{a_d}) \\ &\geq H_{\omega_l(2l)}(0) \\ &\geq H_{\varepsilon_0}(l). \end{aligned}$$

Contradiction!

□

### 7.2.2 Kanamori–McAloon

Recall:

**137 Theorem** ( $\text{KM}_f$ ). *For every  $d, m$  there exists  $R$  such that for every colouring  $C: [R]^d \rightarrow \mathbb{N}$  with  $C(x) \leq f(\min x)$  there exists  $H \subseteq R$  of size  $m$  for which for all  $x, y \in [H]^d$  with  $\min x = \min y$  we have  $C(x) = C(y)$ .*

We will show the transition result for the following variant which is equivalent to  $\text{KM}$  (take  $R$  obtained from applying the previous theorem to  $a + m + 1$ ):

**138 Theorem** ( $\text{KM}_f$ ). *For every  $d, m, a$  there exists  $R$  such that for every colouring  $C: [a, R]^d \rightarrow \mathbb{N}$  with  $C(x) \leq f(\min x)$  there exists  $H \subseteq R$  of size  $m$  for which for all  $x, y \in [H]^d$  with  $\min x = \min y$  we have  $C(x) = C(y)$ .*

**139 Definition.** We denote the least such  $R$  with  $\text{KM}_f^d(a, m)$ .

We have the following estimates from [8]:

**140 Lemma.** *For every  $d \geq 2$  there exists constant  $a_d$  such that*

$$\text{KM}_{i \cdot a_d \cdot m}^d(0, a_d \cdot (m + 1)) > 2_{d-2}(i^m).$$

This implies that for  $i > (a_d \cdot m)^m$  and  $m > c + 2$ :

$$\text{KM}_i^d(0, a_d \cdot (m + 1)) > 2_{d-2}((i + 1)^{c+1}).$$

We denote bad colourings that show this with  $D_i$ .

**141 Theorem.** For all  $d \geq 2$  and  $m > d + c + 3$ :

$$\text{KM}_{\sqrt[c]{\log^{d-2}}}^d(2_{d-2}((a_d \cdot m)^{cm}), a_d \cdot (m + 1) + 1) \geq \text{KM}_{\text{id}}^d(0, m).$$

*Proof:* Given bad colourings  $D: [R]^d \rightarrow \mathbb{N}$  for the identity function we create an intermediate colouring

$$\tilde{C}: [R]^k \rightarrow \mathbb{N} \times \mathbb{N} \times (d + 2) \times (d + 2).$$

Roughly speaking  $\tilde{C}_1$  will be  $D(f(x_1), \dots, f(x_d))$ ,  $\tilde{C}_2$  is  $D_{f(x_1)}$  and  $\tilde{C}_{3,4}$  will ensure that for min-homogeneous sets either  $f(x_1) = \dots = f(x_d)$  or  $f(x_1) < \dots < f(x_d)$  in the manner similar to what we have seen for adjacent Ramsey and Paris–Harrington. We will obtain the required bad colouring by defining  $C$  to be one of the first two coordinates or zero, where the choice is dependent on and coded by the value of the last coordinate. We emphasise again that the lower bound estimates for  $\text{KM}_i^d$  directly influence the functions  $f$  for which we can use this construction.

We take  $f = \sqrt[c+1]{\log^{d-2}}$  and:

$$\tilde{C}(x) = (D(f(x_1), \dots, f(x_d)), D_{f(x_1)}(x), i, j),$$

where  $i$  is the maximal such that  $f(x_1) = \dots = f(x_i)$  and  $j$  is the maximal such that  $f(x_1) < \dots < f(x_j)$  ( $j = 1$  otherwise). Note that  $\tilde{C}_1$  is not everywhere-defined, take it to be 0 if it is undefined (same for  $\tilde{C}_2$ ).

If  $H$  of size at least  $d + 2$  is min-homogeneous for  $\tilde{C}_3$  then the values of this coordinate is 1 or  $d + 1$ . Suppose not, let  $x_1 < \dots < x_{d+1}$  be the first  $d + 1$  elements of  $H$ , then:

$$i = \tilde{C}_3(x_1, x_3, \dots, x_{d+1}) = \tilde{C}_3(x_1, x_2, \dots, x_d) + 1 = i + 1,$$

contradiction.

If  $H$  is min-homogeneous for  $\tilde{C}_4$  then it must by similar argument have values 1, 2 or  $d + 1$ . Let  $x_1 < \dots < x_{d+1}$  be the first  $d + 1$  elements of  $H$ , and suppose

$\tilde{C}_4(x_1, \dots, x_d) = 2$ , then  $f(x_2) = f(x_3)$ , hence  $\tilde{C}_4(x_2, x_3, \dots, x_d) = 1$ , in other words in this case  $\tilde{C}_4$  has value 1 on  $H' = H - \min H$ .

Hence either  $f(x) < f(y)$  for all  $x < y \in H'$  or  $f(x) = f(y)$  for all  $x < y \in H'$ . So  $H'$  is min-homogeneous for  $D$  in the first case, or min-homogeneous for  $D_{\min H'}$  in the latter case.

Encode the last two coordinates into single colouring  $E: [R]^d \rightarrow (d+1)^2$  such that the first of those two cases is encoded in value 0, the latter in 1. We take:

$$C(x) = \begin{cases} (d+1)^2 + 2 \cdot \tilde{C}_1(x) + 1 & \text{if } E(x) = 0 \\ (d+1)^2 + 2 \cdot \tilde{C}_2(x) + 2 & \text{if } E(x) = 1 \\ E(x) & \text{otherwise.} \end{cases}$$

Suppose  $H$  of size greater than  $d+2$  is min-homogeneous for  $C$ , it must have value greater than  $(d+1)^2 + 1$ . Hence  $H' = H - \min H$  is min-homogeneous for either  $D$  or  $D_{f(\min H')}$ . In both cases it has size strictly less than  $a_d \cdot (m+1)$ .

This colouring is  $\sqrt[c]{\log^{d-2}}$  regressive because

$$(d+1)^2 + 2 + 2 \cdot \sqrt[c+1]{\log^{d-2}(x_1)} < \sqrt[c]{\log^{d-2}(x_1)},$$

is ensured by limiting the domain of  $C$  to numbers larger than  $2_{d-2}((a_d \cdot m)^{cm})$ .

□

**142 Corollary.**  $\text{IS}_{d+1} \not\vdash \text{KM}_{\sqrt[c]{\log^{d-2}}}^d$ .

**143 Corollary.**  $\text{PA} \not\vdash \text{KM}_{\log^n}$ .

We sharpen this result:

**144 Theorem.**  $\text{IS}_{d+1} \not\vdash \text{KM}_f^{d+2}$ , where  $f(i) = \sqrt[H_{\omega_{d+2}}^{-1}(i)]{\log^d(i)}$ .

*Proof:* We examine

$$R(m) = \text{KM}_{H_{\omega_d}^{-1}(i) \sqrt{\log^{d-2}}}^d(2_{d-2}((a_d \cdot 2m)^{m \cdot 2m}), a_d \cdot (2m+1) + 1).$$

If  $R(m) \geq H_{\omega_d}(m)$  for infinitely many  $m$  we are finished, so suppose that is not the case. So  $H_{\omega_d}^{-1}(i) \leq m$  for all  $i \leq R(m)$  for all but finitely many  $m$ . Take



nonstandard model  $M$ , use overflow to obtain nonstandard such  $m$ . For this  $m$  we have that

$${}^{H\omega_d^{-1}(i)}\sqrt{\log^{d-2}(i)} \geq \sqrt[m]{\log^{d-2}(i)}$$

for all  $i \leq R(m)$ . Note that the proof of theorem 141 is in  $\text{I}\Sigma_1$ , hence we obtain a nonstandard instance of  $\text{KM}^d$ , which according to theorem 4.4 in [23] implies the existence of an  $I < R(m)$  such that  $I \models \text{I}\Sigma_{d-1} + \neg\text{KM}_f^d$ .

□

**145 Theorem.**  $\text{PA} \not\models \text{KM}_f$ , where  $f(i) = \log^{H\epsilon_0^{-1}(i)}(i)$ .

*Proof:* We examine

$$R(m) = \text{KM}_f^m(2_{m-2}((a_d \cdot 3m)^{m \cdot 3m}), a_d \cdot (3m + 1) + 1).$$

If  $R(m) \geq H_{\epsilon_0}(m)$  for infinitely many  $m$  we are finished, so suppose that is not the case. Take nonstandard model  $M$ , using the same argument as above we obtain a nonstandard instance of  $\text{KM}$  (with nonstandard dimension), which implies again the the existence of an initial segment of  $M$  which is a model of  $\text{PA} + \neg\text{KM}$ .

□



## Chapter 8

# Generalising phase transitions

## 8.1 Conjectures

The general shape of the phase transitions, as remarked at the end of the introduction, suggest an underlying principle to the phenomenon. We state some tentative conjectures expressing this. With the first of these conjectures we state the relationship between the combinatorics involving the constant function and the unprovable versions of the unprovable theorem. Concisely stated we conjecture that such a theorem is unprovable for the inverses of lower bounds estimates, provable for the inverse of an upper bounds estimate.

**146 Conjecture (Threshold).** *Suppose  $T$  is a theory that contains  $\text{IS}_1$ ,  $M_f: \mathbb{N}^2 \rightarrow \mathbb{N}$  is a computable function for all computable  $f$ ,  $M_f \leq M_g$  whenever  $f \leq g$ , and:*

$$\begin{aligned} T &\not\vdash \forall d, x \exists! y M_{\text{id}}(d, x) = y, \text{ but} \\ T &\vdash \forall d, x \exists! y M_k(d, x) = y, \end{aligned}$$

*for every constant function  $k$ . Additionally, suppose that there exist increasing, provably total, functions  $l_c$  and  $u$  such that for every  $c$  there exist  $d, n$  such that for sufficiently large  $k$ :*

$$l_c(k) \leq M_k(d, n) \leq u(k),$$

*then:*

$$\begin{aligned} T &\vdash \forall d, x \exists! y M_{u^{-1}}(d, x) = y, \text{ but} \\ T &\not\vdash \forall d, x \exists! y M_{l_c^{-1}}(d, x) = y, \end{aligned}$$

*for every  $c$ .*

In this thesis we have mostly given a proof of  $T \not\vdash \forall d, x \exists! y M_{l_c^{-1}}(d, x) = y$  using an argument in which we show:

$$M_{l_c^{-1}}(h_1(d, c), h_2(d, c, x)) \geq M_{\text{id}}(d, x),$$

where  $h_1$  and  $h_2$  are primitive recursive.

**147 Example.** Examine  $\text{MDL}_f$  and  $D_k^f$  from Chapter 2. Take  $l_c(i) = i^c$  and  $u(i) = 2^i$ . Assume  $\text{IS}_1 \not\vdash \text{MDL}_{\text{id}}$ . Notice that, due to the pigeonhole principle,  $D_d^k(x) \leq (x + k + 1)^d \leq 2^k$  for  $k$  sufficiently large. Furthermore  $k^c \leq D_{2c+2}^{k+1}(0)$ . Hence if the threshold conjecture is true then  $\text{IS}_1 \vdash \text{MDL}_{\log}$ , but  $\text{IS}_1 \not\vdash \text{MDL}_{\log}$ .

The following conjecture states the sharpening phenomenon which can be observed in all phase transitions.

**148 Conjecture (Sharpening).** *Assuming the conditions in Conjecture 146 and:*

$$\begin{aligned} T &\vdash \forall d, x \exists! y M_{u^{-1}}(d, x) = y, \text{ but} \\ T &\not\vdash \forall d, x \exists! y M_{l_c^{-1}}(d, x) = y, \end{aligned}$$

for every  $c$ ,  $(i, c) \mapsto l_c(i)$  is provably total and  $i \mapsto l_i(i)$  eventually dominates  $u$ . Furthermore, let  $H_n: \mathbb{N} \rightarrow \mathbb{N}$  be an increasing hierarchy of provably total functions such that every provably total function of  $T$  is eventually dominated by an  $H_n$  and let  $H$  be a computable function which eventually dominates every function from this hierarchy. Then:

$$\begin{aligned} T &\not\vdash \forall d, x \exists! y M_f(d, x) = y, \text{ but} \\ T &\vdash \forall d, x \exists! y M_{f_n}(d, x) = y, \end{aligned}$$

where  $f(i) = l_{H^{-1}(i)}^{-1}(i)$  and  $f_n(i) = l_{H_n^{-1}(i)}^{-1}(i)$ .

In this thesis we have always used an ad-hoc argument to show

$$T \not\vdash \forall d, x \exists! y M_f(d, x) = y,$$

involving either using estimates or the model construction obtained for proving the unprovability part for the functions  $l_c^{-1}$ .

**149 Example.** We continue with the earlier example. Notice that  $l_i(i) \geq 2^i$  for  $i \geq 2$ , hence if the sharpening conjecture is true then  $\text{IS}_1 \not\vdash \text{MDL}_f$ , but  $\text{IS}_1 \vdash \text{MDL}_{f_n}$ , where  $f(i) = \sqrt[A^{-1}(i)]{i}$  and  $f_n(i) = \sqrt[A_n^{-1}(i)]{i}$ .

The following approach may yield the unprovability part of the sharpening conjecture:

**150 Conjecture (Sanders<sup>1</sup>).** *Suppose  $\varphi_f$  is a  $\Pi_2$ -sentence for every computable  $f$ ,  $f \geq g$  implies  $\varphi_f \rightarrow \varphi_g$  and:*

$$T \not\vdash \varphi_c,$$

for all  $c$ , then:

$$T \not\vdash \varphi_{H^{-1}},$$

where  $H$  is a provably total function which eventually dominates every provably total function of  $T$ .

---

<sup>1</sup>Communicated by blackboard

Demonstrating this conjecture may involve constructing a nonstandard model using a model in which  $\varphi_c$  is false for some nonstandard  $c$  and one in which  $H^{-1}$  is bounded.

In the case of  $\text{uKM}_f$  we used the fact that the colouring we used to construct a model of PA can be defined entirely in  $\text{IS}_1$  to allow us to also construct it for  $\log^d$ , where  $d$  is some nonstandard  $d \geq H_{\varepsilon_0}^{-1}$  (on the relevant domain and for the case that a specific computable function is bounded by  $H_{\varepsilon_0}$ ). For KM we used:

$$\text{IS}_1 \vdash \forall c (\text{KM}_{\sqrt[c]{\log^d}}^{d+2} \rightarrow \text{KM}_{\text{id}}^{d+2}).$$

When we used recursion theoretic estimates to show  $T \not\vdash \varphi_{f_c}$  we essentially showed:

$$\text{IS}_1 \vdash \forall c (\varphi_{f_c} \rightarrow \varphi_{\text{id}}).$$

A proof of the unprovability parts of the conjectures may require such an extra property.

In these conjectures we have the functions  $H_n$  and  $H$ . The canonical candidates for such functions are derived from the proof theoretic ordinal  $\gamma$  of  $T$ ,  $H$  would in this case be  $H_\gamma$  and  $H_n$  would be  $H_{\gamma[n]}$ . The underlying philosophy of this choice is that the function  $H_\gamma^{-1}$  cannot be proven by  $T$  to be unbounded, hence  $T$  is expected to not be able to distinguish  $\varphi_{H_\gamma^{-1}}$  from  $\varphi_c$ . The choice for  $H_n$  is motivated by the following section.

## 8.2 Upper bounds lemmas

In the proofs of the provability parts of the phase transitions we always used an argument involving upper bounds for the existential witnesses of the theorems with constant parameter. The following lemmas provide an easy tool for reducing such provability proofs to merely providing upper bounds. These lemmas are of interest because they provide the underlying argument of all provability parts of the phase transitions.

**151 Lemma** (Upper bounds lemma). *Suppose  $T$  is a theory that contains  $\text{IS}_1$ ,  $M_f: \mathbb{N}^2 \rightarrow \mathbb{N}$  is a computable function for all computable  $f$  and  $M_f(d, x) \leq M_g(d, x)$  whenever  $f(i) \leq g(i)$  for all  $i \leq M_g(d, x)$ . Additionally, suppose that there exist increasing, provably total, functions  $u, h$  such that for every  $d, n$  and  $k \geq h(d, n)$  we have:*

$$M_k(d, n) \leq u(k),$$

then:

$$T \vdash \forall d, x \exists! y M_{u^{-1}}(d, x) = y.$$

*Proof:* If  $i \leq u(h(d, x))$  then  $u^{-1}(i) \leq h(d, x)$ . Hence:

$$M_{u^{-1}}(d, x) \leq M_{h(d, x)}(d, x) \leq u(h(d, x)).$$

□

**152 Lemma** (Upper bounds sharpening lemma). *Let  $T, M$  be as in the upper bounds lemma. If  $(c, i) \mapsto l_c(i)$  is an increasing provably total function such that there exist provably total  $g_1, g_2$  with  $g_1(d) \leq g_2(d, x)$  for all  $x$  and  $M_k(d, x) \leq l_{g_1(d)}(k)$  whenever  $k \geq g_2(d, x)$ , then:*

$$T \vdash \forall d, x \exists! y M_f(d, x) = y,$$

where  $f(i) = l_{B^{-1}(i)}^{-1}(i)$  and  $B$  is an arbitrary unbounded, increasing and provably total function.

*Proof:* If  $i \leq l_{g_1(d)}(B(g_2(d, x)))$  then:

$$f(i) \leq l_{g_2(d, x)}^{-1}(l_{g_1(d)}(B(g_2(d, x)))) \leq l_{g_2(d, x)}^{-1}(l_{g_2(d, x)}(B(g_2(d, x)))).$$

Therefore:

$$M_f(d, x) \leq M_{B(g_2(d, x))}(d, x) \leq l_{g_1(d)}(B(g_2(d, x))).$$

□

Once upper bounds are known the conditions of these lemmas are easily checked for the natural parametrised theorems which we have examined. The reason for this is that those theorems with constant parameter involve upper bounds at the level of at most the (diagonalised) tower functions, hence such properties as eventual domination involve functions from EFA.

**153 Example.** We examine  $\text{MDL}_f$  again. We already know:

$$D_d^k(x) \leq (x + k + 1)^d \leq k^{d+1} = l_{d+1}(k),$$

whenever  $k \geq 2^d(x + 1)$ . Hence, by the upper bounds sharpening lemma  $\text{IS}_1 \vdash \text{MDL}_{f_n}$ , where  $f_n(i) = {}^{A_n^{-1}(i)}\sqrt{i}$ .

In the following two examples we show the provability part of the the transition results from Chapter 7. We use upper bound estimates from the literature.

**154 Example.** Examine  $\text{PH}_f^d$  with fixed  $d$  and its associated function  $\text{PH}_f^d(m, r)$ . The  $r$  will have the role of  $d$  when applying the upper bounds sharpening lemma. By the Erdős-Rado bounds on Ramsey numbers from [13] if  $k \geq r + m$  then:

$$\text{PH}_k^d(m, r) \leq 2_{d-1}(r^{d^2} \cdot k) + m \leq 2_{d-1}((r^{d^2} + 1) \cdot k) = l_{r^{d^2}+1}(k).$$

Hence, by sharpening,  $\text{IS}_d \vdash \text{PH}_{f_\alpha}^{d+1}$  whenever  $f_\alpha(i) = \frac{\log^d(i)}{H_\alpha^{-1}(i)}$  and  $\alpha < \omega_{d+1}$ .

**155 Example.** Examine  $\text{KM}_f^d$  with fixed  $d$  and its associated function  $\text{KM}_f^d(a, m)$ . The  $m$  will have the role of  $d$  when applying the upper bounds sharpening lemma. We use bounds from Corollary 4.2.3 in [29]:

$$\text{KM}_k^d(a, m) \leq 2_{d-2}(k^{d^2 \cdot m}) + a \leq 2_{d-2}(k^{d^2 \cdot m+2}) = l_{d^2 \cdot m+2}(k),$$

where the second inequality is true for  $k \geq (a + d^2 \cdot m + 2)$ . Hence, by sharpening,  $\text{IS}_{d+1} \vdash \text{KM}_{f_\alpha}^{d+2}$  whenever  $f_\alpha(i) = \sqrt[H_\alpha^{-1}(i)]{\log^d(i)}$  and  $\alpha < \omega_{d+2}$ .

The upper bounds lemmas have conditions which are similar to the upper bounds in the conditions of the transition conjectures. The major difference is that, instead of eventual domination, we have provable eventual domination. This condition may need to be added to the conjectures.





## Chapter 9

# Samenvatting

### 9.1 Onbewijsbare stellingen

De onbewijsbaarheids-, onvolledigheids-, of onafhankelijkheidsleer bestudeert die stellingen die niet bewijsbaar zijn in theorieën van de wiskunde. Al vanaf Gödels onvolledigheidsstellingen is bekend dat zodra een consistente verzameling van axioma's voor de wiskunde ingewikkeld genoeg is om eenvoudige rekenkunde te omvatten, maar nog wel berekenbaar is, er stellingen zullen bestaan die niet bewijsbaar zijn met deze axioma's. Dit resultaat was een antwoord op probleem nummer 2 van de beroemde toespraak van Hilbert in Parijs. In dat probleem werd gevraagd naar een systeem van axioma's voor de wiskunde en een bewijs van consistentie binnen dat systeem.

Een van de kandidaten voor dit programma voor rekenkunde was de Peano rekenkunde (PA). De onvolledigheidsstellingen van Gödel, hoewel zeer interessant, hadden weinig invloed op de wiskunde buiten de logica. De nieuwe vraag werd of er *natuurlijke* rekenkundige stellingen bestaan die niet bewijsbaar zijn in PA. Het eerste voorbeeld kwam in 1977 in de vorm van de volgende stelling van Paris en Harrington:

**1 Stelling.** *Er bestaat voor iedere  $d, c, m$  een  $R$  zodanig dat voor iedere kleuring  $C: [m, R]^d \rightarrow c$  er een  $H \subseteq [m, R]$  met grootte  $\min H$  bestaat zodat  $C$  constant is op  $[H]^d$ .*

Deze stelling is natuurlijk in de zin dat het zeer lijkt op de gewone stelling van Ramsey (het enige verschil is de  $\min H$ , vervang deze door  $m$  om de stelling van Ramsey te bekomen). Sindsdien zijn er vele stellingen opgedoken die niet bewijsbaar zijn in PA. Bovendien bevat deze stelling geen oneindige objecten, dus de onbewijsbaarheid is niet het resultaat van het gebruik van objecten die niet uit te drukken zijn in de taal van PA. Voor verdere voorbeelden verwijzen we naar de Engelse introductie.

## 9.2 Fasenovergangen

Bij de ontwikkeling van niet-bewijsbare stellingen ontstonden er bovendien varianten op deze stellingen die onbewijsbaar bleven. Een voorbeeld hiervan in PA is de volgende stelling van Friedman die gebaseerd is op een oneindige versie van een stelling van Kruskal.

**2 Stelling.** *Er bestaat voor iedere  $l$  een  $K$  zodanig dat voor iedere reeks  $T_0, \dots, T_K$  van bomen, waarbij iedere  $T_i$  ten hoogste  $l+i$  punten bevat, er  $i < j$  bestaan, zodat  $T_i$  in te bedden is in  $T_j$ .*

Door de grenzen op de aantallen punten in de reeksen te beperken verkrijgt men de volgende variaties op de stelling:

**3 Stelling.** *Er bestaat voor iedere  $l$  een  $K$  zodanig dat voor iedere reeks  $T_0, \dots, T_K$  van bomen, waarbij iedere  $T_i$  ten hoogste  $l + r \cdot \log(i)$  punten bevat, er  $i < j$  bestaan, zodat  $T_i$  in te bedden is in  $T_j$ .*

Loebl en Matoušek bewezen dat deze stelling niet bewijsbaar is voor  $r = 4$ , maar wel bewijsbaar voor  $r = \frac{1}{2}$ . Met dit resultaat is de voor de hand liggende vraag bij welke  $r$  er een overgang plaatsvindt van bewijsbaarheid naar onbewijsbaarheid. Het opmerkelijke antwoord hierop, gegeven door Weiermann, is dat deze overgang zit op  $c = \frac{1}{\log(\alpha)}$ , waarbij  $\alpha = 2.9557652865 \dots$  (Otters tree constant).

Bij de stelling van Paris en Harrington was er een vergelijkbaar verschijnsel voor de volgende variant:

**4 Stelling.** *Er bestaat voor iedere  $d, c, m$  een  $R$  zodanig dat voor iedere kleuring  $C: [m, R]^d \rightarrow c$  er een  $H \subseteq [m, R]$  met grootte  $f(\min H)$  bestaat zodat  $C$  constant is op  $[H]^d$ .*

Zoals eerder opgemerkt is dit de bekende bewijsbare stelling van Ramsey als  $f$  een constante functie is en niet bewijsbaar indien  $f$  de identiteitsfunctie is. Het natuurlijke probleem wordt hier het classificeren van de functies  $f$  naar bewijsbaarheid van de verkregen variatie op de stelling van Paris en Harrington. Het antwoord hierop is dat de stelling bewijsbaar is voor  $\log^*$ , maar niet bewijsbaar is voor iedere functie  $\log^n$  (waarbij  $n$  staat voor het aantal iteraties van  $\log$ ). Voor overige voorbeelden verwijzen we naar de Engelse introductie. We bestuderen deze overgangen met het doel een beter begrip te krijgen voor onbewijsbare stellingen.

In deze thesis onderzoeken we voornamelijk het verband tussen deze fasenovergangen en de wiskunde die plaats vindt binnen zeer zwakke deeltheorieën van PA. Er lijkt een verband te bestaan tussen de telproblemen die ontstaan door het beschouwen van de varianten van de stellingen met constante functies en de uiteindelijke fasenovergang.

In het bijzonder bestuderen we in hoofdstuk 6 de adjacent Ramsey stelling van Harvey Friedman. We geven een korte schets.

**5 Stelling** ( $\text{AR}_f$ ). *Er bestaat voor iedere  $k, r$  een  $R$  zodanig dat voor iedere functie  $C: R^k \rightarrow \mathbb{N}^r$ , met  $\max C(x) \leq f(\max x)$  voor alle  $x$ , er  $x_1 < \dots < x_{k+1}$  bestaan, zodat  $C(x_1, \dots, x_k) \leq C(x_2, \dots, x_{k+1})$ .*

Deze stelling is niet bewijsbaar in PA voor  $f = \text{id}$ , maar bewijsbaar voor constante functies. We gebruiken ondergrenzen voor de constante functies van Friedman om aan te tonen dat  $\text{AR}_{\log^n} \rightarrow \text{AR}_{\text{id}}$  voor iedere  $n$ . Het belangrijkste idee hierbij is het combineren van de functie  $C$  met functies  $C_i$  op de volgende manier:

$$D(x) = (C(f(x_1), \dots, f(x_k)), C_{f(x_1)}(x), w(x)).$$

De functies  $C_i$  zijn zodanig gekozen dat  $\max C_i(x) \leq i$  en er bovendien voor  $x_1 < \dots < x_{k+1}$  met  $f(x_1) = \dots = f(x_{k+1}) = i$  geldt dat  $C_i(x_1, \dots, x_k) \not\leq C(x_2, \dots, x_{k+1})$ . De functie  $w$  is zó gekozen dat voor  $x_1 < \dots < x_{k+1}$  met  $f(x_i) = f(x_{i+1}) < f(x_{i+2}) < \dots < f(x_{k+1})$  er geldt dat  $w(x_1, \dots, w(x_k)) \not\leq w(x_2, \dots, x_{k+1})$ .

Als er  $x_1 < \dots < x_{k+1}$  bestaan met  $D(x_1, \dots, x_k) \leq D(x_2, \dots, x_{k+1})$ , dan zorgen bovenstaande eigenschappen ervoor dat  $C(x_1, \dots, x_k) \leq C(x_2, \dots, x_{k+1})$ , wat de implicatie aantoonst. Dit bewijs geeft een verbinding aan tussen de bewijsbare varianten van de stelling en de fasenovergang. We kunnen de functies  $C_i$  namelijk enkel construeren voor parameters  $f$  die ‘inversen’ zijn van de ondergrenzen behorende bij  $\text{AR}_c$  voor constante functies  $c$ .

In Hoofdstuk 7 gebruiken we de zelfde methodiek om de fasenovergangen te bestuderen voor de twee onafhankelijke stellingen van Paris en Harrington en die van Kanamori en McAloon.

Het verband tussen de fasenovergangen en de combinatoriek voor de versies van de stellingen met constante functie is als volgt samen te vatten:

Neem berekenbare functies  $M_f$ , zodanig dat  $M_{\text{id}}$  niet bewijsbaar totaal is in  $T$  en  $(c, i) \mapsto l_c(i)$  wel bewijsbaar totaal. Stel we kunnen voor iedere  $c$  een  $d, x$  vinden zodanig dat:

$$l_c(k) \leq M_k(d, x)$$

voor voldoende grote  $k$ , dan:

$$T \not\vdash \forall d, x \exists! y M_{l_c^{-1}}(d, x) = y.$$

Stel dat:

$$M_k(d, x) \leq u(k)$$

voor voldoende grote  $k$ , dan:

$$T \vdash \forall d, x \exists! y M_{u^{-1}}(d, x) = y.$$

Deze overgangen zijn bovendien verfijnbaar. Neem een stijgende hiërarchie  $H_n$  van stijgende en bewijsbaar totale functies zodanig dat iedere bewijsbaar totale functie van  $T$  eventueel gemajoriseerd wordt door een  $H_n$  en neem een  $H$  die elke functie uit de hiërarchie eventueel majoriseert. Dan:

$$T \not\vdash \forall d, x \exists! y M_f(d, x) = y,$$

waarbij  $f(i) = l_{H^{-1}(i)}(i)$ . Als bovendien  $u$  eventueel door  $i \mapsto l_i(i)$  wordt gemajoriseerd dan:

$$T \vdash \forall d, x \exists! y M_{f_n}(d, x) = y,$$

waarbij  $f(i) = l_{H_n^{-1}(i)}(i)$ .

In hoofdstuk 8 worden deze observaties nader beschouwd.

□

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